9.C Determinant-Functions

As in earlier subsection, let I be a finite indexed set. For a finite dimensional K-vector space V, by 9.B.6 only interesting alternating multilinear maps $V^I \longrightarrow W$ are with $\# I \leq \mathrm{Dim}_K V$. In this subsection, we will study the case $\# I = \mathrm{Dim}_K V$.

9.C.1 Definition Let V be a finite dimensional K-vector space. An alternating I-multi-linear form $V^I \longrightarrow K$ is called a determinant-function on V if $\# I = \mathrm{Dim}_K V$.

Before we describe the determinant function, we mention the following useful Lemma, which follows directly from the general distributive law for multilinear maps:

9.C.2 Lemma Let v_i, $i \in I$, be a system of elements in a K-vector space V. For an alternating multi-linear map $f : V^I \longrightarrow W$ and an I-tuple $(x_j)_{j\in I}$ of linear combinations $x_j = \sum_{i \in I} a_{ij} v_i$, we have:

$$f\left((x_j)_{j\in I}\right) = \sum_{\sigma \in \mathfrak{S}(I)} \left(\mathrm{Sign}\,\sigma \prod_{j\in I} a_{\sigma j, j}\right) f\left((v_j)_{j\in I}\right)$$

Proof It follows from the general distributive law:

$$f\left((x_j)_{j\in I}\right) = \sum_{(i_j)_{j\in I}\in I^I} \left(\prod_{j\in I} a_{i_j,j}\right) f\left((v_{i_j})_{j\in I}\right)$$

Since f is alternating, $f\left((v_{i_j})_{j\in I}\right)=0$ for all $(i_j)_{j\in I}\in I^I$ which are not permutations of I. But, if $(i_j)_{j\in I}$ is a permutation $\sigma\in\mathfrak{S}(I)$, i.e. $\sigma(j)=i_j$ for all $j\in I$, then (by 9.B.4)

$$f\left((v_{i_j})_{j\in I}\right) = f\left((v_{\sigma(j)})_{j\in I}\right) = \operatorname{Sign}\sigma\, f\left((v_j)_{j\in I}\right).$$

Now, the assertion follows by substitution.

The following theorem is the Main Theorem on the Determinant-theory.

9.C.3 Theorem Let $\underline{v}=(v_i)_{i\in I}$ be a basis of the finite dimensional K-vector space V. Then there exists a unique determinant-function $\Delta_{\underline{v}}: V^I \longrightarrow K$ such that $\Delta_{\underline{v}}(\underline{v})=1$. Moreover, for an arbitrary I-tuple $(x_j)_{j\in I}$ with $x_j=\sum_{i\in I} a_{ij}v_i$, $j\in I$, we have:

$$\Delta_{\underline{v}}\left((x_j)_{j\in I}\right) = \sum_{\sigma\in\mathfrak{S}(I)} (\operatorname{Sign}\sigma)\prod_{j\in I} a_{\sigma j,j}$$

$$= \sum_{\sigma\in\mathfrak{S}(I)} (\operatorname{Sign}\sigma)\prod_{j\in I} a_{j,\sigma j}.$$

Proof The uniqueness and the first equality directly follows from 9.C.2.

For a proof of uniqueness, let v_i^*, $i \in I$, be the dual basis of $\underline{v}$ and let $f := \bigotimes_{i\in I} v_i^*$ be the product of v_i^*, $i \in I$, i.e. the map $f : V^I \longrightarrow K$ defined by $f((x_i)_{i\in I}) := \prod_{i\in I} v_i^*(x_i)$. Then by 9.B.7 the map

$$\Delta_{\underline{v}} := Af = \sum_{\sigma\in\mathfrak{S}(I)} (\operatorname{Sign}\sigma)\, \sigma f$$

is alternating I-multilinear and

$$\Delta_{\underline{v}}((x_j)_{j\in I}) = \sum_{\sigma\in\mathfrak{S}(I)} (\operatorname{Sign}\sigma) \prod_{i\in I} v_i^*(x_{\sigma i})$$

$$= \sum_{\sigma\in\mathfrak{S}(I)} (\operatorname{Sign}\sigma) \prod_{i\in I} a_{i,\sigma i}$$

In particular, $\Delta_{\underline{v}}((v_j)_{j\in I}) = 1$ and $\Delta_{\underline{v}} = Af$ is the required determinant-function.

The equality of the both representations of $\Delta_{\underline{v}}((x_j)_{j\in I})$ in 9.C.3 also follows directly from the equalities:

$$\prod_{j\in I} a_{\sigma j, j} = \prod_{i\in I} a_{i,\sigma^{-1} i} \quad \text{and} \quad \operatorname{Sign}\sigma = \operatorname{Sign}\sigma^{-1}$$

for every permutation $\sigma \in \mathfrak{S}(I)$.

9.C.4 Corollary Let $\underline{v} = (v_i)_{i\in I}$ be a basis of the finite dimensional K-vector space V and let $\Delta_{\underline{v}} : V^I \longrightarrow K$ be the corresponding determinant-function as in 9.C.3. Then for every I-multi-linear alternating map $f : V^I \longrightarrow W$ and every I-tuple $(x_j)_{j\in I} \in V^I$, we have:

$$f\left((x_j)_{j\in I}\right) = \Delta_{\underline{v}}\left((x_j)_{j\in I}\right) f\left((v_j)_{j\in I}\right)$$

<u>Proof</u> The assertion follows directly from 9.C.2 and 9.C.3.

Therefore, with the notations as in 9.C.4, the map $f \longmapsto f\left((v_j)_{j\in I}\right)$ from the space $\mathrm{Alt}(I; V, W)$ of alternating I-multi-linear maps $V^I \longrightarrow W$, onto W, is an isomorphism. In particular,

<u>9.C.5 Corollary</u> Let $\underline{v} = (v_i)_{i\in I}$ be a basis of the finite dimensional K-vector space V. Then $\Delta_{\underline{v}}$ is a basis of the 1-dimensional K-vector space of the determinant-functions $V^I \longrightarrow K$.

We apply 9.C.3 to the K-vector space K^I and the standard basis $e_i, i\in I$. The elements of K^I are interpreted as column-vectors. An I-tuple $(x_j)_{j\in I}$ of vectors $x_j \in K^I, j \in I$, is therefore an $I\times I$-matrix $\mathfrak{A} = (a_{ij}) \in M_I(K)$ whose columns are $x_j = \sum_{i \in I} a_{ij} e_i$, $j \in I$.

Now, let $\Delta_{\underline{e}}$ be the determinant-function corresponding to the standard basis $\underline{e} = (e_i)_{i \in I}$ of K^I as in 9.C.3 and therefore $\Delta_{\underline{e}}(\underline{e}) = 1$. This function $\Delta_{\underline{e}}$ is called the <u>standard-determinant function</u> on K^I. By 9.C.3 we have $\Delta_{\underline{e}}((x_j)_{j \in I}) = \operatorname{Det} \mathfrak{A}$, where the determinant $\operatorname{Det} \mathfrak{A}$ of the matrix $\mathfrak{A} \in M_I(K)$ is defined as follows:

<u>**9.C.6 Definition**</u> For a square $I \times I$ matrix $\mathfrak{A} = (a_{ij}) \in M_I(K)$, the element (of K)

$$|\mathfrak{A}| := \operatorname{Det} \mathfrak{A} := \sum_{\sigma \in \mathfrak{S}(I)} \operatorname{Sign} \sigma \prod_{j \in I} a_{\sigma j, j}$$

$$= \sum_{\sigma \in \mathfrak{S}(I)} \operatorname{Sign} \sigma \prod_{i \in I} a_{i, \sigma i}$$

is called the <u>determinant</u> of the matrix $\mathfrak{A}$. The function $\operatorname{Det} : M_I(K) \longrightarrow K$, $\mathfrak{A} \longmapsto \operatorname{Det} \mathfrak{A}$, is called the <u>determinant-function</u> on $M_I(K)$.

The determinant $\operatorname{Det} \mathfrak{E}_I$ of the unit-matrix $\mathfrak{E}_I \in M_I(K)$ is 1. From the equality of the both sum-formulas for $\operatorname{Det} \mathfrak{A}$, it follows that the determinant does not change by transition to the transpose of matrix. Therefore:

9.C.7 Theorem For every matrix $\mathfrak{A} \in M_I(K)$, we have $\operatorname{Det} \mathfrak{A} = \operatorname{Det} {}^t\mathfrak{A}$.

In the situation of 9.C.3, we have:

$$\Delta_{\underline{v}}\left((x_j)_{j\in I}\right) = \operatorname{Det}\left(a_{ij}\right) \text{ for } (x_j)_{j\in I} \in V^I,$$

where $x_j = \sum_{i\in I} a_{ij} v_i$, $j \in I$. In particular:

9.C.8 Theorem Let $\underline{v} = (v_i)_{i\in I}$ and $\underline{v}' = (v_i')_{i\in I}$ be two bases of the finite dimensional K-vector space V and let $\mathfrak{B} = (b_{ij}) \in GL_I(K)$ be the transition matrix from $\underline{v}$ to $\underline{v}'$, i.e.

$$v_j = \sum_{i\in I} b_{ij} v_i', \quad j \in I. \text{ Then}$$

$$\Delta_{\underline{v}'} = (\operatorname{Det} \mathfrak{B}) \cdot \Delta_{\underline{v}} \quad \text{or} \quad \Delta_{\underline{v}} = (\operatorname{Det} \mathfrak{B})^{-1} \Delta_{\underline{v}'}.$$

Proof It follows that

$$\Delta_{\underline{v}'}\left((v_j)_{j\in I}\right) = \operatorname{Det} \mathfrak{B} = (\operatorname{Det} \mathfrak{B})\, \Delta_{\underline{v}}\left((v_j)_{j\in I}\right)$$

and hence $\Delta_{\underline{v}'} = (\operatorname{Det} \mathfrak{B}) \cdot \Delta_{\underline{v}}$.

9.C.9 Theorem Let $\Delta : V^I \longrightarrow K$ be a non-zero determinant-function. Then $(x_i)_{i\in I} \in V^I$ is a basis of V if and only if $\Delta\left((x_i)_{i\in I}\right) \neq 0$.

Proof If $(x_i)_{i\in I}$ are linearly dependent, then $\Delta\left((x_i)_{i\in I}\right) = 0$ by 9.B.5. Now, let $\underline{x} = (x_i)_{i\in I}$ be a basis of V. Then $\Delta = a\,\Delta_{\underline{x}}$ with $a = \Delta((x_i)_{i\in I})$ (by 9.C.5). Since $\Delta \neq 0$, it follows that $a \neq 0$ also.

9.C.10 Corollary Let $\mathfrak{A} = (a_{ij}) \in M_I(K)$ be a square $I\times I$ matrix over K. Then $\mathfrak{A}$ is invertible if and only if $\operatorname{Det}\mathfrak{A} \neq 0$.

Proof Note that $\mathfrak{A}$ is invertible if and only if the columns x_j, $j\in I$, of $\mathfrak{A}$ form a basis of K^I. Now, since $\operatorname{Det}\mathfrak{A} = \Delta_{\underline{e}}((x_j)_{j\in I})$, the assertion follows from 9.C.9.

The rank of a matrix can also be charactersied by using determinants. We consider an $m\times n$ matrix

$$\mathfrak{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

For subsets $I = \{i_1,\dots,i_r\}$ and $J = \{j_1,\dots,j_r\}$ of $\{1,\dots,m\}$ and $\{1,\dots,n\}$ respectively with $1\le i_1<\dots<i_r\le m$ and $1\le j_1<\dots<j_r\le n$, the determinant of the $r\times r$-submatrix

$$\mathfrak{A}_{I,J} = \mathfrak{A}(i_1,\dots,i_r;j_1,\dots,j_r) := \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_r} \\ \vdots & \ddots & \vdots \\ a_{i_r,j_1} & \cdots & a_{i_r,j_r} \end{pmatrix}$$

of $\mathfrak{A}$, is called a <u>r-minor</u> or a <u>minor of order r</u> of $\mathfrak{A}$. The diagonal-submatrices $\mathfrak{A}_{I,I} = \mathfrak{A}(i_1,\dots,i_r;i_1,\dots,i_r)$ are simply denoted by $\mathfrak{A}_I = \mathfrak{A}(i_1,\dots,i_r)$. Minors of the type $\operatorname{Det} \mathfrak{A}(i_1, \dots, i_r)$ are called diagonal-minors and the determinants of the matrices $\mathfrak{A}(1, \dots, r)$, $r \le \operatorname{Min}(m,n)$ are called principal-minors.

9.C.11 Minor-Criterion for the Rank Let $\mathfrak{A} \in M_{m,n}(K)$ be a matrix of rank r ($\le \operatorname{Min}(m,n)$). Then all minors of an order $> r$ of $\mathfrak{A}$ are 0 and there exists a minor of order r of $\mathfrak{A}$ which is not 0.

Proof Let $s > r$ and $1 \le i_1 < \dots < i_s \le m$, $1 \le j_1 < \dots < j_s \le n$. The submatrix

$$\mathfrak{A}(i_1, \dots, i_s; j_1, \dots, j_s) = \begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_1, j_s} \\ & & \\ a_{i_s, j_1} & \cdots & a_{i_s, j_s} \end{pmatrix}$$

of $\mathfrak{A}$, has rank at most r and hence is not invertible. Therefore, its determinant is 0 and hence all minors of order $> r$ of $\mathfrak{A}$ are 0.

There exist r linearly independent columns of $\mathfrak{A}$, say of indices, $j_1 < \dots < j_r$. The matrix (submatrix of $\mathfrak{A}$) with these columns is then has the maximal rank r. Among the m rows of this matrix, there are r linearly independent rows, say of indices, $i_1 < \dots < i_r$. Then the matrix $\mathfrak{A}(i_1, \dots, i_r; j_1, \dots, j_r) =$

$$\begin{pmatrix} a_{i_1, j_1} & \cdots & a_{i_1, j_r} \\ \vdots & \ddots & \vdots \\ a_{i_r, j_1} & \cdots & a_{i_r, j_r} \end{pmatrix}$$

is invertible and hence its determinant (which is a r-minor of $\mathfrak{A}$) is not 0 by 9.C.10.

<u>9.C.12 Example</u> Let $K \subseteq L$ be an extension of fields and let $\mathfrak{A} \in M_{m,n}(K)$. From 9.C.11 it follows that the rank of $\mathfrak{A}$ considered as matrix over K or as matrix over L are equal. This was already applied in Exercise