E0 221 Discrete Structures / August-December 2012

(ME, MSc. Ph. D. Programmes)

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

TA/Corrections by : **Dr. Anita Das** (anita@csa.iisc.ernet.in)

1-st Midterm : Saturday, September 22, 2012; 10:00 -12:00 **2-nd Midterm :** Saturday, October 27, 2012; 10:00 -12:00
Final Examination : December ??, 2012, 10:00 -13:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	>90	76–90	61–75	46–60	35–45	<35

1. Sets[1] – Operations on Sets

1.1 For a set A, show that the following statements are equivalent:

(i) $A=\emptyset$.
(ii) $A\setminus B=A\cap B$ for every set B.
(iii) There exists a set B with $A\setminus B=A\cap B$.
(iv) $B\setminus A=B\cup A$ for every set B.
(v) There exists a set B with $B\setminus A=B\cup A$.

1.2 For sets A and B, show that the following statements are equivalent:

(i) $A\subseteq B$.
(ii) $A\cap B=A$.
(iii) $A\cup B=B$.
(iv) $A\setminus B=\emptyset$.
(v) $B\setminus(B\setminus A)=A$.
(vi) $A\cup(B\cap C)=(A\cup C)\cap B$ for every set C.
(vii) There exists a set C with $A\cup(B\cap C)=(A\cup C)\cap B$.

1.3 Show that the operation $(\cdot\setminus\cdot)$ is distributive over the operations $\cup$ and $\cap$, i. e. for arbitrary three sets A,B,C :

(a) $(A\cup B)\setminus C=(A\setminus C)\cup(B\setminus C)$.

(b) $(A\cap B)\setminus C=(A\setminus C)\cap(B\setminus C)$.

1.4 Show that $(A\cap B)\cap(A\setminus B)=\emptyset$ and $(A\cap B)\cup(A\setminus B)=A$.

1.5 For sets A,B,C, show that:

[1] Although sets were not formally introduced into mathematics until the late nineteenth century, they now form the foundation on which most of mathematics is built. Set theory is the proper framework for abstract mathematical thinking and it may be developed axiomatically. The aim of this section is to provide sufficient familiarity with the notation and terminology of set theory to enable us to state definitions and theorems in set-theoretic language. The first person to realize the importance of sets as a mathematical subject for study was the German mathematician Georg Cantor (1845-1918). Cantor's first paper on the use of sets, published in 1874, was very controversial because it was innovative and differed with the thinking of the time. To understand the trouble surrounding Cantor's set theory, one must understand the social climate of the time. Most scientist were conservative and proud of the successes of science. Many scientist went so far as to say "Most physical principles have already been uncovered with the exception of a few refinements here and there." The powerful Leopold Kronecker (1823-1891) attacked Cantor's set theory as more theology than mathematics. Although Kronecker's animosity toward Cantor was based on scientific fact, more often than not the attack became petty and degenerated into a personal vendetta. The hypersensitive Cantor finally could take no more. At the age of 40, he suffered a mental breakdown and briefly entered a mental institution at Halle. Although he made many of his major contributions to set theory after that time, he suffered nervous breakdowns for last 23 years of his life. Cantor died despondent in a mental institution in 1918 at the age of 74.

(a) $A \setminus (B \cup C) = (A \setminus B) \setminus C$.
(b) $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C) = (A \cap C) \setminus B$.
(c) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.
(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
(e) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
(f) $A \cup (B \setminus C) \supseteq (A \cap B) \setminus (A \cap C)$. Further, show that the equality holds if and only if $A \cap C = \emptyset$ and give an example to show that, in general, the equality does not hold.

1.6 Prove that the statements $A \subseteq B \iff (C \setminus B) \subseteq (C \setminus A)$ and $(C \setminus (C \setminus A)) = A$ are not true for some sets A, B and C.

1.7 For sets A, B, C, show that:

(a) $A \triangle A = \emptyset$ and $A \triangle \emptyset = A$.	(b) $A = B$ if and only if $A \triangle B = \emptyset$.
(c) $A \cap B = \emptyset$ if and only if $A \triangle B = A \cup B$.	(d) $(A \triangle B) \cap (A \cap B) = \emptyset$.
(e) $(A \triangle B) \cup (A \cap B) = A \cup B$.	(f) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.
(g) If $A \triangle B = A \triangle C$, then $B = C$.	(h) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(**Remark :**)

1.8 For non-empty sets A and B, show that the following statements are equivalent:
(i) $A \times B \subseteq B \times A$. (ii) $B \times A \subseteq A \times B$. (iii) $A \times B = B \times A$. (iv) $A = B$.

(**Remark :** This shows that $\times$ is not a commutative operation.)

1.9 Show that the operation $\times$ is distributive over the operations $\cup$, $\cap$ and $(\cdot \setminus \cdot)$, i. e. for arbitrary three sets A, B, C :
(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(B \cup C) \times A = (B \times A) \cup (C \times A)$.
(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ and $(B \cap C) \times A = (B \times A) \cap (C \times A)$.
(c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ and $(B \setminus C) \times A = (B \times A) \setminus (C \times A)$.

1.10 Let A, B, C and D be sets. If $A \subseteq C$ and $B \subseteq D$, then show that $A \times B \subseteq C \times D$. Moreover, if $A \neq \emptyset$ and $B \neq \emptyset$, then the converse also holds.

1.11 For sets A, B, C, D, show that:
(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(b) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. Moreover, the equality holds if and only if the conditions "$A \subseteq C$ or $D \subseteq B$" and "$C \subseteq A$ or $B \subseteq D$" are satisfied.
(c) If $A \subseteq C$ and $D \subseteq B$, then show that $(C \times D) \setminus (A \times B) = ((C \setminus A) \times D) \cup (C \times (D \setminus B))$.

1.12 Let A be a set and let $\mathfrak{P}(A)$ be the power set of A. Show that
(a) If $B \in \mathfrak{P}(A)$, then $\mathfrak{P}(B) \subseteq \mathfrak{P}(A)$. **(b)** $\bigcap_{B \in \mathfrak{P}(A)} B = \emptyset$.

(c) Let I be a set whose members are also sets. Then

$$\bigcap_{A \in I} \mathfrak{P}(A) = \mathfrak{P}(\cap_{A \in I} A) \text{ and } \bigcup_{A \in I} \mathfrak{P}(A) \subseteq \mathfrak{P}(\cap_{A \in I} A).$$

Moreover, the last inclusion is, in general, not an equality.

Below one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

Though the naive or intuitive approach to sets will suffice for most purpose, an exposition of general set theory requires more precision, for without explicit axioms, it is well known that various contradictions arise. It was the famous English philosopher Bertrand Russell[2] (1872-1970) who shook the Mathematics community in 1901 by declaring the admission of a set of all sets would lead to a contradiction. This is the famous Russell's Paradox[3]:

T1.1 Russell's Paradox: Serious difficulties occur if we allow the notion of set to be too general. For example, it is undesirable to talk of "the set U of all sets." If such a set exists, it must, being a set, be a member of itself, that is, $U \in U$. This is not in itself disastrous, but now consider Russell's famous set V of *all sets which are not members of themselves.* (Russell's argument is to be compared with the ancient paradoxes of the "liar" type, which were the subject of innumerable commentaries in classical formal logic; the question whether a man who says "I am lying" is telling the truth or not when he speaks these words.) If V is not a member of itself then by that fact it qualifies as a member of V, that is, it is then a member of itself and the situation is no better if we yield to this reasoning and allow that V is a member of itself. For then V does not qualify as a member of V, which only admits as members those sets which are not members of themselves, so we find that we have again proved the opposite of what we have assumed.

Therefore, we must accept the terms "set" and "element" as undefined terms or primitives and guide these primitives by a number of axioms. It is desired to indicate only a framework within which we will work, which avoids the known antinomies and which, at least until now, has not led to any contradiction.

In what follows, we are assuming that we are working with *classical logic*. To prevent any misunderstanding, ambiguity or arbitrary interpretation, the essential definitions as well as the axioms of the theory of sets are introduced below using logical connectives:

(1) $\vee$ ("or") (in the sense "one or the other *or both*"),

(2) $\neg$ ("not"),

(3) $\exists$ ("there exists" or "for some"),

(4) ($\wedge$ ("and"),

(5) $\forall$ ("for every"),

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[2]Bertrand Russell was born on May 18, 1872, at Trelleck, Wales. before he was four both of his parents died. He had been a shy, silent boy until he entered Trinity Collge, Cambridge University in 1880. After three years of Mathematics he concluded that what he was being taught was full of errors. He sold of his Mathematics books and changed to philosophy. In his *Pricipia Mathematica* (1910-1913), a three volume monumental work co-authored with Alfred North Whitehead (1861-1947), he attempted to recast set theory so as to avoid paradoxes. In 1918 he wrote, "I want to stand at time rim of the world and peer into the darkness beyond and see a little more than others have seen .... I want to bring back into world of men some little bit of wisdom." He certainly did, more than just "some little bit." In the same year he was put into prison for an unfavorable comment about the American Army. In 1950 he received the Order of Merit from King of England and the Nobel prize for literature. In his later years he led a number of demonstrations against nuclear warfare.

[3]Russell's Paradox was not the only one to arise in set theory. Shortly after the Russell's Paradox appeared many paradoxes were constructed by several mathematicians and logicians. This precipitated a search for a rigorous foundation of set theory which would avoid contradictions. As a consequence of all these paradoxes, many mathematicians and logicians (with David Hilbert (1862-1943) among its leaders) have contributed to several brands such as "*Zermelo-Fraenkel-Skolem axiomatic set theory*" (proposed by Ernst Friedrich Ferdinand Zermelo (1871-1953), Adolf Abraham Halevi Fraenkel (1891-1965) and Thoralf Albert Skolem (1887-1963)) and "*von Neumann-Bernays-Gödel axiomatic set theory*." (proposed by John von Neumann (1903-1957), Paul Isaac Bernays (1888-1977) and Kurt Gödel (1906-1978)) of "axiomatic set theory," each designed to avoid these paradoxes and at the same time to preserve the main body of Cantor's set theory.

(6) $\Rightarrow$ ("implies"),

(7) $\iff$ ("if and only if"), but we are rather casual about this.

A set is a collection of well-defined objects[4] – meaning that if A is a set and a is some object then either a is definitely in A or a is definitely not in A. The objects that form the set are called elements or members of the set. We denote sets by capital letters A, B, C, etc. and elements of sets are denoted by lower case letters a, b, c etc. If a is an element of a set A then we write $a \in A$ and read it as "a belongs to A." If a is not an element of a set A then we write $a \notin A$ and read it as "a does not belong to A."

For the sake of completeness, below we list axioms of set theory which guarantees existence of at least one set and postulates that certain constructions using sets will yield sets.

T1.2 (Axiomatic Set Theory) We recall here the following six axioms of the *Zermelo-Fraenkel-Skolem axiomatic set theory*, namely: *Extensionality*, *Replacement*, *Power-Set*, *Sum-Set*, *Infinity* and *Choice*.

(1) (Axiom of Extensionality) Two sets A and B are equal if and only if they have exactly the same elements, i. e. every element of A is an element of B and conversely. In symbols: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. (**Remarks :** In the Theory of Sets the equality $=$ is reflexive, symmetry and transitive.)

(2) (Axiom of Replacement) For every set A and every *binary predicate* $F(x,y)$ which is *functional in x*, there exists a unique set whose elements are precisely the *mates* of all elements of A with respect to $F(x,y)$. (**Remarks :** One of the most important consequences of the axiom of replacement is the following:

Theorem of Separation: *For every set A and every monadic predicate $P(x)$, there exists a unique set whose elements are precisely all those elements of A that satisfy $P(x)$.*

A significant application of the theorem of separation which asserts the existence of an *empty set* (i. e. a set having no elements. The theorem below justifies the introduction of the symbol $\emptyset$:

Theorem *There exists a unique empty set and the empty set is a subset of every set.*

Another application of the theorem of Separation, let us show that, from the existence of the set $\{a,b,c\}$, where a,b,c are distinct, we can derive the existence of its subsets $\{a\}$ and $\{a,b\}$. For this consider the predicates $(x \neq b) \wedge (x \neq c)$ and $(x \neq c)$, respectively. Then $\{a\} = \{x \in \{a,b,c\} \mid (x \neq b) \wedge (x \neq c)\}$ and $\{a,b\} = \{x \in \{a,b,c\} \mid (x \neq c)\}$.)

(3) (Axiom of Power-Set) For every set A, the set whose elements are precisely all subsets of A exists. This set is called the Power-Set of A and is usually denoted by $\mathfrak{P}(A)$. (**Remarks :** In view of the axiom of Power-Set, from the existence of $\emptyset$ (the empty set), we assert the existence of $= \mathfrak{P}(\emptyset) = \{\emptyset\}$. Similarly, the sets $\mathfrak{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$, $\mathfrak{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ exist. Further, the above three sets are distinct. Therefore in view of the axioms of Extensionality, Replacement and Power-Set, we can assert that many more distinct sets exist.)

(4) (Axiom of Sum-Set) For every set I, the set whose elements are precisely the elements of all elements of I exists. This set is called the sum-set of I or the union of the elements of I and is usually denoted by $\bigcup_{A \in I} A$.

(5) (Axiom of Infinity) There exists a set W such that:

(a) $\emptyset \in W$. **(b)** If $x \in W$, then $x^+ := x \cup \{x\} \in W$.

[4] For example, one should never say "consider the set A of some students from IISc who are registered for the Discrete Structures course". For it is not definite whether "John$\in A$" or John$\notin A$. However, since every positive integer is definitely either a prime number or not a prime number, one can consider the set $\mathbb{P}$ of all positive prime numbers. It may be hard to determine whether a given object is in a set. For example, it is unknown whether $2^{2^{17}} + 1$ is in the set $\mathbb{P}$. However, it is certainly either prime or not prime.

For $x \in W$, the element x^+ is called the immediate successor of x. (**Remarks :** A set W satisfying (a) and (b) above is called an inductive set. Clearly some of the elements of W are: $\emptyset$, $\emptyset \cup \{\emptyset\} = \{\emptyset\}$, $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

We shall prove that:

Theorem *There exists a unique inductive set* $\mathbb{N}$ *which is 'minimal".* i. e. *if W is any other inductive set, then $N \subseteq W$.*

Proof

This set is called the set of all natural numbers and its elements are called natural numbers. Accordingly, we introduce the following familiar notation for the elements of $\mathbb{N}$: $0 = \emptyset$, $1 = \{\emptyset\} = \{0\} = 0^+$, $2 = \{\emptyset, \{\emptyset\}\} = 1 \cup \{1\} = \{0,1\} = 1^+$, $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = 2 \cup \{2\} = \{0,1,2\} = 2^+$, $4 = 3 \cup \{3\} = \{0,1,2,3\} = 3^+$, $5 = 4 \cup \{4\} = \{0,1,2,3,4\} = 4^+$. What is worth noting is that *every natural number is a set*.

The set of natural numbers $\mathbb{N}$ is a set with unlimited number of elements in the sense that one cannot finish counting its elements. By means of axioms of Replacement, Power-Set and Infinity, we may infer the existence of more and more extensive sets. For instance, from the set $\mathbb{N}$, in view of the axioms of Power-Set and Replacement, we derive the existence of the set $I = \{\mathbb{N}, \mathfrak{P}(\mathbb{N}), \mathfrak{P}(\mathfrak{P}(\mathbb{N})), \ldots\}$ from which, by the axiom of Sum-Set, we infer the existence of the set $\bigcup_{A \in I} A$ which is rather extensive set.)

(6) (Axiom of Choice) For every disjointed set I with $\emptyset \notin I$, there exists a choice-set C of I. (**Remarks :** Let us recall that a set C is called a choice-set or selection-set of a set $I = \{A \mid A \in I\}$ if for every element A of I the set C has exactly one element that belongs to A. Clearly, $C \subseteq \bigcup_{A \in I} A$. If $\emptyset \in I$, then I has no choice-set. On the other hand the empty set $I = \emptyset$ has only one choice set namely the empty set $\emptyset$. Two sets A and B are called disjoint if they have no common elements, i. e. if $A \cap B = \emptyset$. A set I is called disjointed if its distinct elements are pairwise disjoint.)

(7) (Axiom of Regularity)

T1.3 (Summary)

The logical impasse can be avoided by restricting the notion of set, so that "very large" collections or the "collection of all things" are not counted as sets. We shall never need to deal with any sets large enough to cause trouble in this way and consequently we may put aside all such worries and hope that paradoxes will not appear.

However, our main interest in this course is the application of the theory of sets to the basic notions of mathematics. For example, in formulating fundamental notions such as relations, functions, natural numbers, integers, rational numbers, real numbers, ordinal numbers and cardinal numbers as well as their arithmetic [5]

Therefore we shall take a naive, non-axiomatic approach of set theory. In fact the entire discussion may be made rigorously precise.

T1.4 (Operations on Sets) We collect some important operations os sets. Let A, B and C be arbitrary sets.

(1) The set B is called a subset of A (or A is called a supset of B) if every element of B is an element of A. The notations $B \subseteq A$ or $A \supseteq B$ are both used to mean "B is a subset of A." or "A is a supset of B". The symbol $\subseteq$ is called the inclusion of the set B in in the set A. Moreover, if $B \neq A$, then we also write $B \subset A$ or $B \subsetneq A$ and say that B is a proper subset of A (or A is a proper supset of B). (**Remarks :** The inclusion relation $\subseteq$ is not to be confused with the

[5] The arithmetic of ordinal and cardinal numbers is also called transfinite arithmetic.

membership relation $\in$. For example, $\emptyset \subseteq \emptyset$, but not $\emptyset \in \emptyset$; $\{\emptyset\} \in \{\{\emptyset\}\}$ but $\{\emptyset\} \not\subseteq \{\{\emptyset\}\}$ because there is a member of $\{\emptyset\}$, namely, $\emptyset$, that is not a member of $\{\{\emptyset\}\}$; Let US be the set of all people in the United States and let UN be the set of all countries belonging to the United Nations. Then John Jones $\in$ US $\in$ UN, but John Jones $\in$ UN (since he is not even a country), and hence US $\not\subseteq$ UN.)

(2) There is exactly one set with no elements and is called the e m p t y (or n u l l or v a c u - o u s)- s e t which is usually denoted[6] by $\emptyset$ and is a subset of every set.

(3) (U n i o n) There is a unique set $A\cup B$ such that $x\in A\cup B$ if and only if either $x\in A$ or $x\notin B$. In symbols: $A\cup B := \{x \mid x\in A \text{ or } x\in B\}$ and is called the u n i o n of the sets A and B. (**Remark :** By repeating this operation we can form the union of three sets, four sets etc. moreover, form the union of finitely many sets. But suppose we want to form the union of *infinitely many* sets, then we need a more general union operation. This leads us to the following definition: For any set I (whose members are sets), the set of all the elements of all the members of I is called the u n i o n - s e t or the s u m - s e t of I. In symbols: in the case that $I = \{A \mid A\in I\}$, the union-set of I $\{x \mid x\in A \text{ for some } A\in I\}$ is usually denoted by $\bigcup_{A\in I} A$. For example, if $I = \{\emptyset, \{\emptyset\}\}$, then $\bigcup_{A\in I} A = \{\emptyset\} \neq \{\emptyset,\{\emptyset\}\}$. However, $\bigcup_{A\in\emptyset} A = \bigcup_{A\in\{\emptyset\}} A = \emptyset$. The operation $\cup$ is idempotent, commutative and associative, see Test-Exercise T1.5.)

(4) (I n t e r s e c t i o n) There is a unique set $A\cap B$ such that $x\in A\cap B$ if and only if $x\in A$ and $x\in B$. In symbols: $A\cap B := \{x \mid x\in A \text{ and } x\in B\}$ and is called the i n t e r s e c t i o n of the sets A and B. (**Remark :** As in the case of union, we also need the corresponding generalization of the intersection operation. In general, we define for every non-empty set I, the subset of $\bigcup_{A\in I} A$ consisting of all common elements of all the members of I is called the i n t e r s e c t i o n - s e t of I and is denoted by $\bigcap_{A\in I} A$. In symbols: $x\in \bigcap_{A\in I} A$ if and only if $x\in A$ for every $A\in I$. In contrast to the union operation, there is no special axiom is needed to justify the intersection operation. However, there is one trouble extreme case, namely, what happens if $I=\emptyset$?. There is no set C such that for every x, $x\in C$ if and only if x belongs to every member of $\emptyset$, since the right hand side is true for every x. This presents a mild notational problem: How to define $\bigcap\emptyset$? One option[7] is to leave $\bigcap\emptyset$ undefined, since there is no very satisfactory way of defining it!

For example, if $I = \{\emptyset,\{\emptyset\}\}$, then $\bigcap_{A\in I} A = \{\emptyset\} \neq \{\emptyset,\{\emptyset\}\}$. The operation $\cap$ is idempotent, commutative and associative, see Test-Exercise T1.5. For non-emptyset I, clearly $\bigcap_{A\in I} A \subseteq \bigcup_{A\in I} A$. Further, note that the union-set $\bigcup_{A\in I} A$ of the set $I = \{A \mid A\in I\}$ is the "smallest" set which includes all the sets $A\in I$ and the intersection-set $\bigcap_{A\in I} A$ of the set I is the "largest" set which is a subset of every set $A\in I$.)

(5) (D i f f e r e n c e) There is a unique set $A\setminus B$ such that $x\in A\setminus B$ if and only if $x\in A$ and $x\notin B$. In symbols: $A\setminus B := \{x \mid x\in A \text{ and } x\notin B\}$ and is called the d i f f e r e n c e of the sets A and B. If B is a subset of A, then the difference set $A\setminus B$ is also called the c o m p l e m e n t o f B i n A and is usually denoted by $\complement_A B$. (**Remark :** Note that for every set A, $A\setminus\emptyset = A$, $\emptyset\setminus A = \emptyset$ and $A\setminus A=\emptyset$. Therefore the difference operation $(\cdot\setminus\cdot)$ is not commutative. Further, since $(A\setminus\emptyset)\setminus A = \emptyset$ and $A\setminus(\emptyset\setminus A) = A$, it is also not associative. It is interesting to note that the inclusion and intersection can be expressed in terms of the difference: $A\subseteq B$ if and only if $A\setminus B=\emptyset$; and $A\cap B = A\setminus(A\setminus B)$.)

(6) (S y m m e t r i c D i f f e r e n c e) The s y m m e t r i c d i f f e r e n c e $A\triangle B$ of the sets A and B is the set of all those elements that are elements of A or B but not of both. In symbols: $A\triangle B = (A\cup B)\setminus(A\cap B)$. Clearly, $A\triangle B = \{x \mid \text{ either } x\in A \text{ or } x\in B\} = (A\setminus B)\cup(B\setminus A)$ which justifies the term the symmetric difference. (**Remark :** The symmetric difference operation is commutative and nilpotent, i. e. $A\triangle B = B\triangle A$ and $A\triangle A = \emptyset$. Moreover, it is associative, see also Exercise 1.??.)

(7) (P o w e r - S e t) There is a unique set $\mathfrak{P}(A)$ whose elements are precisely all subsets of A. This set is called the p o w e r - s e t of A. Note that[8] unions, intersections, differences and

[6]The symbol $\emptyset$ is not the Greek letter phi ϕ, but rather a letter of Danish and Norwegian alphabets. The symbols $\square$ and $\wedge$ also appear in literature for $\emptyset$.

[7]This option works perfectly well, but some logicians dislike it. It leaves the intersection-set $\bigcap_{A\in\emptyset} A$ of the empty-set $\emptyset$ as an untidy loose end, which they may later trip over!

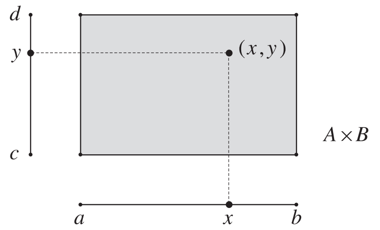
[8]Therefore $\cup$, $\cap$, $(\cdot\setminus\cdot)$ and $\triangle$ are binary operations on the power-set $\mathfrak{P}(A)$.

symmetric differences of the sets from $\mathfrak{P}(A)$ are again members of $\mathfrak{P}(A)$. For example, the power-set $\mathfrak{P}(\{1,2,3\})$ of the set $\{1,2,3\}$ is the set $\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. Note that the power-set $\mathfrak{P}(\emptyset)=\{\emptyset\}$ of the empty-set $\emptyset$ is a non-empty set.

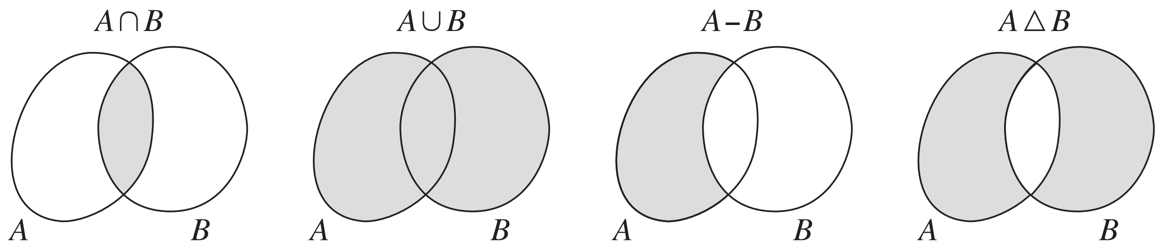
(8) (Cartesian-product) For sets A and B, the set of (ordered) pairs (a,b), $a\in A$ and $b\in B$, is called the Cartesian-product or the cross-product of the sets A and B and is usually denoted by $A\times B$. (**Remark :** The set $\{\{a\},\{a,b\}\}$ is called the ordered pair and is denoted by (a,b). The two ordered pairs (a,b) and (a',b') are equal if and only if the corresponding components are equal, i. e. $a=a'$ and $b=b'$. Therefore the pair (a,b) and the set $\{a,b\}$ needs to be distinguished! For $a\neq b$, naturally $(a,b)\neq(b,a)$ but $\{a,b\}=\{b,a\}$.)

For example, let $A:=[a,b]=\{x\in\mathbb{R}\mid a\le x\le b\}$ and $B:=[c,d]=\{y\in\mathbb{R}\mid c\le y\le d\}$ be two closed intervals with $a<b$ and $c<d$. Then $A\times B$ is the "rectangle":

$$[a,b]\times[c,d]=\{(x,y)\in\mathbb{R}\times\mathbb{R}\mid a\le x\le b\,,\,c\le y\le d\}\,.$$



T1.5 (Euler-Venn Diagrams[9]) Euler-Venn diagrams are drawings that illustrate abstract ideas. Generally circles are drawn so that they overlap to illustrate set theory concepts.



T1.6 (Algebra of Sets[10] and Computational-Rules) The following identities which hold for arbitrary sets, are some computational rules in the algebra of sets:

Let A,B and C be arbitrary sets. Then:

(1) $A\cup\emptyset=A$ and $A\cap\emptyset=\emptyset$.

(2) $A\cup A=A$ and $A\cap A=A$. (Idenpotency)

(3) $A\cup B=B\cup A$ and $A\cap B=B\cap A$. (Commutativity)

(4) $A\cup(B\cup C)=(A\cup B)\cup C$ and $A\cap(B\cap C)=(A\cap B)\cap C$. (Associativity)

(5) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ and $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$. (Distributivity)

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[9] John Venn (1834-1923) was an English logician who made contributions to logic and probability. He was an ordained minister but resigned his ministry in 1883 to concentrate on logic, which he taught at Cambridge. The diagrams for which he remembered were actually used earlier by a Swiss mathematician Leonhard Euler (1707-1783), but were perfected by Venn.

[10] The study of operations of union $\cup$, intersections $\cap$, differences $(\cdot\setminus\cdot)$ together with the inclusion $\subseteq$ goes by the name algebra of sets. In some ways algebra of sets obeys laws reminiscent of algebra of real numbers (with $+,\cdot,-$ and $\le$), but there are significant differences!

(6) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. (De Morgan's laws[11])

T1.7 For arbitrary three sets A, B, and C, show that:
(a) $(A \setminus B) = A$ if and only if $(B \setminus A) = B$.
(b) $B = \emptyset$ if and only if $A \cup B = A \setminus B$.
(c) $A = B$ if and only if $(A \setminus B) = (B \setminus A)$.
(d) $A \subseteq B \cup C$ if and only if $(B \setminus C) \subseteq A$.
(e) $(B \setminus A) \subseteq C$ if and only if $(B \setminus A) = B$.

T1.8 For arbitrary four sets A, B, C and D, show that:
(a) $(A \setminus B) \cup C = ((A \setminus (B \setminus C)) \cup (C \setminus A)$.
(b) $A \cup (B \setminus C) = ((A \cup B) \setminus C) \cup (A \cap C)$.
(c) $(A \setminus B) \cup C = ((A \cup C) \setminus B) \cup (B \cap C)$.
(d) $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$.
(e) $(A \setminus B) \setminus (C \setminus D) = (A \setminus (B \cup C)) \cup ((A \cap D) \setminus B)$.
(f) $A \setminus (B \setminus (C \setminus D)) = (A \setminus B) \cup ((A \cap C) \setminus D)$.
(g) $A \setminus (A \setminus (B \setminus (B \setminus C))) = A \cap B \cap C$.
(h) $(A \setminus D) \subseteq (A \setminus B) \cup (B \setminus C) \cup (C \setminus D)$.

[11] Augustus De Morgan (1806-1871) was an English logician who made major contributions to logic and probability. De Morgan was a brilliant mathematician who introduced the slash notation for representing fractions, such as $1/2$ and $3/4$. Once asked when he was born, De Morgan replied, "I was x years old in the year x^2." Can you determine the year he was born ?