E0 221 Discrete Structures / August-December 2012

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

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1-st Midterm : Saturday, September 22, 2012; 10:00 -12:00 **2-nd Midterm :** Saturday, October 27, 2012; 10:00 -12:00
Final Examination : December ??, 2012, 10:00 -13:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

2. Relations

2.1 Let R and S be relations from a set X to a set Y.

(a) Show that $R \cup S$ is a relation from X to Y and compute $\text{Dom}(R \cup S)$ and $\text{Rng}(R \cup S)$.

(b) Construct relations R and S from X to Y such that $\text{Rng}(R \cap S) \neq \text{Rng}(R) \cap \text{Rng}(S)$.

(c) Express $(R \cup S)^{-1}$ and $(R \cap S)^{-1}$ in terms of R^{-1} and S^{-1}.

2.2 Let R and S, T be relations from X to Y and from Y to Z, respectively. Does the equality $(T \cup S) \circ R = (T \circ R) \cup (S \circ R)$ holds? What happens if union is replaced by intersection?

2.3 Let $I := [0,1] := \{x \in R \mid 0 \le x \le 1\}$ be a closed interval in $\mathbb{R}$ and let $R := \{(x,y) \in I \times I \mid x < y\}$. Draw the pictures of $I \times I$, R, δ_I, $R \cup \Delta_I$ and $R \cap \Delta_I$.

2.4 Let X be a non-empty set. A non-empty subset $\mathfrak{A}$ of the set of all relations $\text{Rel}(X)$ is called an u n i f o r m i t y on X if the following properties hold:

(1) If $R \in \mathfrak{A}$, then $\Delta_X \subseteq R$.

(2) If $R \in \mathfrak{A}$ and $S \in \mathfrak{A}$, then $R \cap S \in \mathfrak{A}$.

(3) If $R \in \text{Rel}(X)$ and $S \in \mathfrak{A}$ with $S \subseteq R$, then $R \in \mathfrak{A}$.

(4) If $R \in \mathfrak{A}$, then $R^{-1} \in \mathfrak{A}$.

(5) If $R \in \mathfrak{A}$ and $S \in \mathfrak{A}$, then $S \circ R \in \mathfrak{A}$.

For each positive real number $r > 0$, let $R_r := \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| < r\}$ and let

$$\mathfrak{A} := \{S \in \text{Rel}(\mathbb{R}) \mid R_r \subseteq S \text{ for some positive real numebr } r\}.$$

Show that $\mathfrak{A}$ is a uniformity on $\mathbb{R}$.

Below one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

Roughly speaking, a "relation" is a rule for assigning to certain sets certain other sets. For example, to each natural number greater than 2 each natural number greater than it.

T2.1 (Relations) Let X and Y be sets. A (binary) relation[1] R from X and Y is a subset $R \subseteq X \times Y$, i.e. an element $R \in \mathfrak{P}(X \times Y)$. For the expression "$(x,y) \in R$" we shall write "$xRy$" and say that "*$x$ is related to y with respect to R*", $x \in X$, $y \in Y$. The set of relations $\mathfrak{P}(X \times Y)$ from X to Y is also denoted by $\text{Rel}(X,Y)$ and its elements are also denoted by the symbols $\sim$, $\cong$ $\equiv$, $\leq$, $\preceq$ $\cdots$. In the case $Y = X$, we put $\text{Rel}(X) = \text{Rel}(X,X) = \mathfrak{P}(X \times X)$ and its elements are called relation on X.

Let R be a relation form a set X to a set Y. The subset $\{x \in X \mid \text{there exists } y \in Y \text{ such that } (x,y) \in R\}$ of X of all first coordinates of R is called the domain of R and is usually denoted by $\text{Dom}(R)$. The subset $\{y \in Y \mid \text{there exists } x \in X \text{ such that } (x,y) \in R\}$ of Y of all second coordinates of R is called the range or image of R and is usually denoted by $\text{Rng}(R)$ or $\text{Im}(R)$. In particular, $R \subseteq \text{Dom}(R) \times \text{Rng}(R)$, but this inclusion may be strict (see parts (c) and (d) below)
Let X and Y be sets.

(a) The empty set (called the empty relation) $\emptyset$ is a relation from X to Y and $\text{Dom}(\emptyset) = \emptyset = \text{Rng}(\emptyset)$. The product $X \times Y$ is also a relation from X to Y and if $X \times Y \neq \emptyset$, then $\text{Dom}(X \times Y) = X$ and $\text{Rng}(X \times Y) = Y$.

(b) The diagonal subset $\Delta_X := \{(x,x) \mid x \in X\}$ is a relation on X and is called the diagonal or identity relation on X.

(c) The subset $\{(x,A) \in X \times \mathfrak{P}(X) \mid x \in A\}$ is a relation from X to $\mathfrak{P}(X)$, called the elementhood relation, its domain is X and range is $\mathfrak{P}(X) \setminus \{\emptyset\}$.

(d) The subset $\{(A,B) \in \mathfrak{P}(X) \times \mathfrak{P}(X) \mid A \subseteq B\}$ is a relation on $\mathfrak{P}(X)$, called the inclusion relation.

(e) The subset $< := \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ is a relation, called the strict order relation on $\mathbb{N}$. Further, $\text{Dom}(<) = \mathbb{N}$ and $\text{Rng}(<) = \mathbb{N} \setminus \{0\}$. In particular, $< \subsetneq \text{Dom}(<) \times \text{Rng}(<)$.

(f) If S are relation form X to Y and if $R \subseteq S$, then R is also a relation from X to Y and $\text{Dom}(R) \subseteq \text{Dom}(S)$ and $\text{Rng}(R) \subseteq \text{Rng}(S)$. In particular, if R and S are relations from X to Y, then $R \cap S$ is also relation from X to Y and $\text{Dom}(R \cap S) \subseteq \text{Dom}(R) \cap \text{Dom}(S)$ and $\text{Rng}(R \cap S) \subseteq \text{Rng}(R) \cap \text{Rng}(S)$.

T2.2 (Composite and Inverse relations) Let R be a relation from X to Y. Then the subset

$$R^{-1} := \{(y,x) \in Y \times X \mid (x,y) \in R\} \subseteq Y \times X$$

is a relation from Y to X, called the inverse (or opposite or reverse) of the relation R and is usually denoted by R^{-1}.
Further, if S be a relation from Y to Z. Then the subset

$$S \circ R := \{(x,z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\} \subseteq X \times Z$$

is a relation from X to Z, called the composition (or composite or product) relation and is usually denoted[2] by $S \circ R$.

[1] More generally, for every positive integer n, one can define n-ary relation as a subset of $X^n := X \times \cdots \times X$ (n-times). We shall rarely consider n-ary relation for $n \neq 2$ and so by relation from now on we shall mean a binary relation unless otherwise specified.

[2] Despite of our habit of reading/writing from left to right, the definition of $S \circ R$ suggests "first R, then S." Although, it might therefore seem natural to denote the composition of S and R by $R \circ S$, the notation adopted here is justified by its traditional usage in the case of relations which are "functional."

For example, $(\Delta_X)^{-1} = \Delta_X$, $(<)^{-1} = \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n > m\}$ is the inverse relation of the strict order relation on $\mathbb{N}$ and $(X \times Y)^{-1} = Y \times X$. The inverse relation of the constant relation $R_c := \{(x,c) \mid x \in X\}$, $c \in Y$ is $R_c^{-1} = \{(c,x) \mid x \in X\}$.

(1) Let R be a relation from X to Y and let S be a relation from Y to Z. Then

(a) $\mathrm{Dom}(R^{-1}) = \mathrm{Rng}(R)$, $\mathrm{Rng}(R^{-1}) = \mathrm{Dom}(R)$ and $(R^{-1})^{-1} = R$.

(b) $\mathrm{Dom}(S \circ R) = \mathrm{Dom}(R)$, $\mathrm{Rng}(S \circ R) = \mathrm{Rng}(S)$ and $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

(c) $\Delta_{\mathrm{Dom}(R)} \subseteq R^{-1} \circ R$ and $\Delta_{\mathrm{Rng}(R)} \subseteq R \circ R^{-1}$.

(2) Let R, S and T be relations from X to Y, from Y to Z and from Z to W, respectively. Then

(a) (Associativity of composition) $T \circ (S \circ R) = (T \circ S) \circ R$.

(b) $R \circ \Delta_X = R = \Delta_Y \circ R$.