E0 221 Discrete Structures / August-December 2012

(ME, MSc. Ph. D. Programmes)

Download from : `http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...`

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

TA/Corrections by : Dr. Anita Das (anita@csa.iisc.ernet.in)

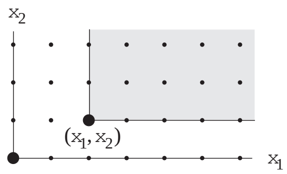
1-st Midterm : Saturday, September 22, 2012; 10:00 -12:00 **2-nd Midterm :** Saturday, October 27, 2012; 10:00 -12:00
Final Examination : December ??, 2012, 10:00 -13:00

Evaluation Weightage : Midterms (Two) : 50% **Final Examination :** 50%

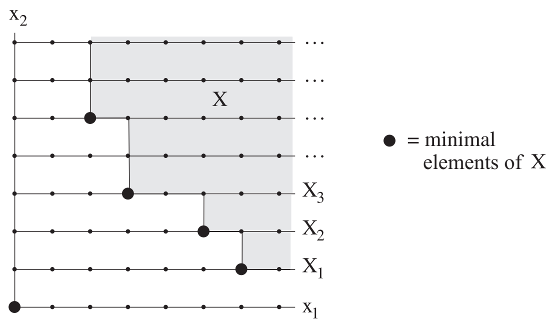
Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

3. Ordered Sets

*__3.1__ Let $r \in \mathbb{N}^+$. The set $\mathbb{N}^r = \mathbb{N} \times \cdots \times \mathbb{N}$ (r-times) of the r-tuples of natural numbers is ordered by the product order of the usual order on $\mathbb{N}$, i. e. by definition $(x_1,\ldots,x_r) \leq (y_1,\ldots,y_r)$ if and only if $x_i \leq y_i$ for all $i = 1,\ldots,r$. For $r \geq 2$, this product order is not a total order. For $r = 2$, the set of points $\geq (x_1,x_2)$ is shaded in the following picture:



Sketch the picture for the set of points $\leq (x_1,x_2)$. Clearly $(0,\ldots,0) \in \mathbb{N}^r$ is the least element of $\mathbb{N}^r$. Let X be an arbitrary subset of $\mathbb{N}^r$ (with the induced order from the product order of $\mathbb{N}^r$). Show that X has only finitely many minimal elements. (**Hint :** Proof by induction on r. For this one may assume that if a r-tuple $x \in X$, then all r-tuples $y \in \mathbb{N}^r$ with $x \leq y$ also belong to X. This means replace X by the set $\bigcup_{x \in X} (x + \mathbb{N}^r)$. Note that this does not change the set of minimal elements. For the inductive step from r to $r+1$, apply induction-hypothesis to the sets $X'_n := \{x' \in \mathbb{N}^r \mid (x',n) \in X\} \subseteq \mathbb{N}^r$, $n \in \mathbb{N}$. Observe that $X'_0 \subseteq X'_1 \subseteq X'_2 \subseteq \cdots$ and there exists $n_0 \in \mathbb{N}$ such that $X'_n = X'_{n_0}$ for all $n \geq n_0$. This already proves the case from $r = 1$ to $r+1 = 2$:



and makes it clear how can one make a general argument.)

3.2 On the set $\mathbb{N}^+$ of positive natural numbers, let $\mid$ denote the relation "divides", i. e. for $m,n \in \mathbb{N}^+$, $m \mid n$ if and only if $n = am$ for some $a \in \mathbb{N}^+$. Show that:

(a) $\mid$ is an order on $\mathbb{N}^+$ and that the element 1 is the least element.

(b) The prime numbers are precisely the minimal elements in $(\mathbb{N}^+ \setminus \{1\}, |)$.

(c) Draw the Hasse-Diagrams for the set of divisors of 12 and 30.

(d) The chains in $(\mathbb{N}^+, |)$ are either finite or an infinite sequence of the type

$$\mathscr{C} = (q_0,\ q_0q_1,\ q_0q_1q_2,\ \dots\ )$$

with $q_n \in \mathbb{N}^+$, $q_n \geq 2$ for all $n \geq 1$. Moreover, $\mathscr{C}$ is a maximal chain if and only if the sequence $(q_0,\ q_1,\ q_2,\ \dots\ )$ is infinite and $q_0 = 1$ and q_n, $n \geq 2$ are all prime numbers.

3.3 **(a)** Let I be any set and let $\mathfrak{P}(I)$ be the power set of I. The natural inclusion $\subseteq$ defines and order on $\mathfrak{P}(I)$. Moreover, the ordered set $(\mathfrak{P}(I), \subseteq)$ is a complete ordered set.

(b) The set of natural numbers $\mathbb{N}$ with the usual order $\leq$ is a conditionally complete ordered set. In this ordered set there are no lower cuts. For each $n \in \mathbb{N}$, the ordered set $(\{0,1,\dots,n\}, \leq)$ is a complete ordered set.

(c) Give an order $\preceq$ on the set $\mathbb{N}$ of natural numbers so that $(\mathbb{N}, \preceq)$ is a complete ordered set.

3.4 Give an order $\preceq$ on the set $\mathbb{N} \times \mathbb{N}$ so that:

(a) The ordered set $(\mathbb{N} \times \mathbb{N}, \preceq)$ has infinitely many lower cuts.

(b) The ordered set $(\mathbb{N} \times \mathbb{N}, \preceq)$ has no lower cuts.

3.5 Let $(X, \leq)$ be an dense ordered set and let a and b be two elements of X with $a < b$. Prove that there exist infinitely elements distinct elements $x \in X$ such that $a < x < b$, i. e. the open interval (a,b) of X is infinite.

3.6 Let $(X, \leq)$ be a simply ordered set.

(a) Show that X is dense if and only if no element of X has an immediate successor.

(b) Let $x, y \in X$ and $I(x)$ be an initial segment of X. Then show that:

(i) If the initial segment $I(y)$ of X is a subset of $I(x)$, then $I(y)$ is also an initial segment of $I(x)$.

(ii) If $I(y)$ is an initial segment of $I(x)$, then $I(y)$ is also an initial segment of X.

(c) Give an example of a simply ordered set $(X, \leq)$, such that every initial segment of which contains only finitely many elements.

3.7 **(a)** Well order the set of natural numbers $\mathbb{N}$ so that exactly five of its elements do not have predecessors.

(b) Give three well orders on the set $\mathbb{Z}$ of integers.

(c) Give a well order on the set $\mathbb{Q}$ of rational numbers..

3.8 Let X and Y be well ordered sets. Let $\leq$ defined by:

For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$: $(x_1, y_1) \leq (x_2, y_2)$ if and only if $y_1 < y_2$ or $y - 1 = y_2$ and $x_1 \leq x_2$.

(a) Show that $\leq$ is a well order on $X \times Y$.

(b) Is the *dictionary order* on $X \times Y$ a well ordering?

Below one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

Relations which order a set occur in all domains of mathematics and in many branches of the empirical sciences. There are almost an endless number of interesting theorems about various ordering relations and their properties.

T3.1 (Orders)
An element $b \in X$ is called an immediate successor of $a \in X$ if $a < b$ and there is no element $c \in X$ such that $a < c < b$. If b is an immediate successor of a, then a is called an immediate predecessor of b.

T3.2 Let $(X, \leq)$ be a finite non-empty ordered set. Show that X has (at least one) a minimal and (at least one) a maximal element. Further, prove that X is simply ordered if and only if X is well ordered.

T3.3 Let $\mathbb{N}$ be the set of natural numbers. Show that each of the relations $\preceq$ defined below on the set $\mathbb{N}$ of natural numbers are orders: For $m, n \in \mathbb{N}$, define $m \preceq n$ if

(a) $m \leq n$ (in the usual sense) and m and n have the same parity.

(b) m is even, and m and n have different parity.

(c) $m = n$ or upon the division of m and n by 5, m yields the smaller remainder.

(d) $m = n$ or in the division in (c), m and n yield the same remainder and $m < n$.

T3.4 For the sake of convenience in giving many examples , we shall use the following configuration such as:

$$\{\, \ldots\, ,\, (\, \ldots ; a; b;\, \ldots\, ; c;\, \ldots\,)\, ,\, (\, \ldots ; m; n;\, \ldots\,)\, , \ldots\, \}$$

to represent the ordered set $(X, \leq)$, where $X = \{\, \ldots\, , a, b, c,\, \ldots,\, , m, n,\, \ldots\, \}$ and where $x \leq y$ if and only if x and y are contained in the same parentheses and x is not written to the right of y.

For example, in this notation the set of natural numbers (respectively of integers) with the usual order $\leq$ is represented as: 5

$$(\mathbb{N}, \leq) = \{\, (0; 1; 2;\, \ldots\,)\} \quad (\text{respectively} \quad (\mathbb{Z}, \leq) = \{\, (\, \ldots\, ; -2; -1; 0; 1; 2;\, \ldots\,)\}\,).$$

T3.5 Give an example of an ordered set $(X, \leq)$ such that:

(a) there are exactly three minimal elements and two maximal elements and neither a minimum nor a maximum.

(b) Every non-empty subset Y of X has a least upper bound, but not necessarily a lower bound.

(c) Every non-empty bounded above subset has a least upper bound, but not every subset has a lower bound.

T3.6 Let $(X, \leq)$ be an ordered set.

(1) (Dual) The inverse (or opposite) relation $\leq^{-1}$ (or $\leq^{\mathrm{op}}$) on X is again an order on X. Moreover $(\leq^{-1})^{-1} = (\leq^{\mathrm{op}})^{\mathrm{op}} = \leq$. The partially ordered set $(X, \leq^{\mathrm{op}})$ is called the dual (or opposite) of $(X, \leq)$. Clearly $(X, \leq)$ and $(X \leq^{\mathrm{op}})$ are duals of each other.

(2) (Distinguished elements) It is clear that a distinguished element – a *minimal*, a *maximal*, the *least* and the *greatest* element in $(X, \leq)$ becomes its counterpart (or dual) – a *maximal*, a *minimal*, the *greatest* and the *least* element in the dual ordered set $(X, \leq^{\mathrm{op}})$. Moreover, a *lower bound* (the *subset* $\mathrm{LB}_X(Y)$ *of lower bounds*), an *upper bound* (the *subset* $\mathrm{UB}_X(Y)$ *of upper bounds*), the *greatest lower bound* ($\mathrm{GLB}_X(Y)$) and the *least upper bound* ($\mathrm{LUB}_X(Y)$) of a subset Y of X in $(X,\leq)$ becomes its d u a l – an *upper bound* (the *subset* $\mathrm{UB}_{X^{\mathrm{op}}}(Y)$ *of upper bounds*), a *lower bound* (the *subset* $\mathrm{LB}_{X^{\mathrm{op}}}(Y)$ *of lower bounds*), the *least upper bound* ($\mathrm{GLB}_{X^{\mathrm{op}}}(Y)$) and the *greatest lower bound* ($\mathrm{LUB}_{X^{\mathrm{op}}}(Y)$) of the subset Y of X in the dual ordered set $(X,\leq^{\mathrm{op}})$.

(3) (D u a l i t y T h e o r e m) *Let a statement* $\mathscr{T}$ *be a theorem in every ordered set and let* $\mathscr{T}^{\mathrm{dual}}$ *be the statement* (called the dual of $\mathscr{T}$) *obtained from the statement by changing every distinguished element to its dual and* $\leq$ *to* $\leq^{\mathrm{op}}$. *Then the statement* $\mathscr{T}^{\mathrm{dual}}$ *is again a theorem in every ordered set.* (**Proof** Since the distinguished elements of X as well as those of a subset Y of X in every ordered set $(X,\leq)$ become their duals in the dual ordered set $(X,\leq^{\mathrm{op}})$ and the ordered sets $(X,\leq^{\mathrm{op}})$ and $(X,\leq)$ are duals of each other, the assertion is immediate.)

T3.7 (D u a l T h e o r e m s) We give some concrete examples of the dual theorems.

(1) $\mathscr{T}$: *An ordered set* $(X,\leq)$ *has a least element if and only if the empty-set* $\emptyset$ *has a least upper bound in* X. (**Remark :** Note that $\mathscr{T}$ is a theorem in every ordered set $(X,\leq)$: For a proof note that $\mathrm{UB}_X(\emptyset)=X$ and hence $\mathrm{Min}(X)=\mathrm{LUB}_X(\emptyset)=\mathrm{Min}(\mathrm{UB}_X(\emptyset)$.)

The dual of $\mathscr{T}$ is the following:

$\mathscr{T}^{\mathrm{dual}}$: *An ordered set* $(X,\leq)$ *has a greatest element if and only if the empty-set* $\emptyset$ *has a greatest lower bound in* X.

(2) $\mathscr{T}$: *Let* $(X,\leq)$ *be an ordered set and let* $Y\subseteq X$ *and* $z:=\mathrm{LUB}_X(\mathrm{LB}_X(Y))$. *Then*

$$\mathrm{LUB}_X(\mathrm{LB}_X(Y))\in\mathrm{LB}_X(Y)\quad\textit{and}\quad\mathrm{LUB}_X(\mathrm{LB}_X(Y))=\mathrm{GLB}_X(Y)\,.$$

(**Remark :** Note that $\mathscr{T}$ is a theorem in every ordered set $(X,\leq)$: For a proof note that $z\leq y$ for every $y\in Y$ and hence $z\in\mathrm{LB}_X(Y)$. On the other hand $x\leq z$ for every $x\in\mathrm{LB}_X(Y)$. Therefore $z=\mathrm{GLB}_X(Y)$.)

The dual of $\mathscr{T}$ is the following:

$\mathscr{T}^{\mathrm{dual}}$: *Let* $(X,\leq)$ *be an ordered set and let* $Y\subseteq X$. *Then*

$$\mathrm{GLB}_X(\mathrm{UB}_X(Y))\in\mathrm{UB}_X(Y)\quad\textit{and}\quad\mathrm{GLB}_X(\mathrm{UB}_X(Y))=\mathrm{LUB}_X(Y)\,.$$

T3.8 Theorem *Let* $(X,\leq)$ *be an ordered set. Then every non-empty bounded above subset* Z *of* X *has a least upper bound if and only if every non-empty bounded below subset* Y *of* X *has a greatest lower bound.* In other words the following two statements are equivalent :

(i) $\mathrm{LUB}_X(Z)$ *exists for every bounded above subset* Z *of* X.

(ii) $\mathrm{GLB}_X(Y)$ *exists for every bounded below subset* Y *of* X.

(**Proof** (i)$\Rightarrow$(ii): Let Y be a non-empty bounded below subset of X. Then $\mathrm{LB}_X(Y)\neq\emptyset$ and is bounded above by every element of Y. Therefore $\mathrm{LUB}_X(\mathrm{LB}_X(Y))$ exists by (i) and by $\mathrm{LUB}_X(\mathrm{LB}_X(Y))=\mathrm{GLB}_X(Y)$. The converse (implication (ii)$\Rightarrow$(i)) follows from the duality Theorem T3.6 (3).)

T3.9 (C o n d i t i o n a l l y C o m p l e t e O r d e r e d S e t s) An ordered set $(X,\leq)$ in which every non-empty subset that is bounded above has a least upper bound (and therefore, in which every non-empty subset that is bounded below has a greatest lower bound) is called a c o n d i - t i o n a l l y c o m p l e t e ordered set.

T3.10 As a corollary of the above Theorem in T3.8 we have: *Let* $(X,\leq)$ *be an ordered set. Then every subset* Z *of* X *has a least upper bound if and only if every subset* Y *of* X *has a greatest lower bound.* In other words the following two statements are equivalent :

(i) $\mathrm{LUB}_X(Z)$ *exists for every subset* Z *of* X.

(ii) $\mathrm{GLB}_X(Y)$ *exists for every subset* Y *of* X.

T3.11 (Complete Ordered Sets) An ordered set $(X,\leq)$ in which every subset has a least upper bound (and hence, in which every subset has a greatest lower bound) is called a complete ordered set or a complete lattice. Clearly, a complete lattice has a greatest ($\operatorname{Inf}\emptyset$) and a least ($\operatorname{Sup}\emptyset$) element. Further, every complete ordered set is conditionally complete.

T3.12 (Lower Cuts) Let $(X,\leq)$ be an Ordered Set. A subset L of X is called a lower cut of X if

(i) $\emptyset \neq L \subsetneq X$, i. e. L is a non-empty proper subset of X.

(ii) L has no greatest element, i. e. $\operatorname{Max} L$ does not exist.

(iii) If $x \in L$, then every element $y \in X$ with $y \leq x$ is also an element of L.

With this definition we have the following:

Theorem *Let $\mathscr{L}(X)$ be the set of lower cuts of an ordered set $(X,\leq)$. Then the ordered set $(\mathscr{L}(X),\subseteq)$ is conditionally complete. Moreover, if $(X,\leq)$ is simply ordered, then $(\mathscr{L}(X),\subseteq)$ is also simply ordered.*

T3.13 Let $(X,\leq)$ be a conditionally complete ordered set and $f: X \to X$ be a non-decreasing map from X to X. If there are $a,b \in X$ such that $a \leq f(a) \leq f(b) \leq b$, then there exists an element $c \in X$ such that $a \leq c \leq b$ and $f(c) = c$. In particular, f has a fixed point.

T3.14 Let $(X,\leq)$ be a complete ordered set and $f: X \to X$ be a non-decreasing map from X to X. Then f has at least one fixed point. (**Hint :** Since X is complete, it has a least and greatest elements, put $a := \operatorname{Min} X$ and $b := \operatorname{Max} X$. Then $a \leq f(a) \leq f(b) \leq b$ and hence we can apply Test-Exercise T3.14.)

T3.15 (Initial Segments) Let $(X,\leq)$ be an ordered set. For every element $a \in X$, the set $I(a) := \{x \in X \mid x < a\}$ is called the initial segment of X determined by a. If $I(a) \neq \emptyset$, then $I(a)$ is called a proper initial segment of X.

With this definition we have the following:

(1) Proposition *Let $(X,\leq)$ be an ordered set and let $I(b)$ be an initial segment of an element $b \in X$. If $I(b) \neq \emptyset$, then $I(b)$ is a lower cut of X.*

(2) Theorem *Let $\mathscr{I}(X)$ be the set of all initial segments of an ordered set $(X,\leq)$. If $(X,\leq)$ is simply ordered, then the ordered set $(\mathscr{I}(X),\subseteq)$ is again simply ordered.*

T3.16 (Dense and Continuous Ordered Sets)

(1) An ordered set $(X,\leq)$ is called dense if for every two elements $a,b \in X$ with $a < b$, there exists an element $c \in X$ such that $a < c < b$.

(2) An ordered set $(X,\leq)$ is called continuous if it is simply ordered, dense and conditionally complete.

With this definition we have the following:

(3) Theorem *Let $(X,\leq)$ be a simply ordered set. Then X is dense if and only if every non-empty initial segment of X is a lower cut of X.*

(4) Theorem *Let $(X,\leq)$ be a continuous ordered set and let $L \subseteq X$. Then L is a lower cut of X if and only if L is a proper initial segment of X.*

(5) Theorem *Let $\mathscr{L}(X)$ be the set of lower cuts of an ordered set $(X,\leq)$. If X is simply ordered and dense, then the ordered set $(\mathscr{L}(X),\subseteq)$ is continuous.*

T3.17 Let $(X,\leq)$ be an ordered set.

(a) Let $\mathscr{I}(X)$ be the set of initial segments of X. Suppose that X is dense. Is the ordered set $(\mathscr{L}(X), \subseteq)$ dense?

(b) Let $\mathscr{L}(X)$ be the set of lower cuts of X. Suppose that X is simply ordered. Is the ordered set $(\mathscr{I}(X), \subseteq)$ conditionally complete?

(c) Give an example of an ordered set $(X, \leq)$ such that the ordered set $(\mathscr{I}(X), \subseteq)$ of initial segments of X is complete and simple ordered.

T3.18 Let $(W, \leq)$ be an well ordered set.

(1) Every element in W other than the last element has a unique immediate successor. Moreover, every element of W has at most one immediate predecessor.

(2) Every well ordered set is conditionally complete.

(3) The ordered set $(\mathscr{I}(W), \subseteq)$ of initial segments of W is well ordered.

T3.19 One of the significant features of well ordered set , $(W, \leq)$ is that *if every element of an initial segment $I(x)$ of W satisfies a property implies that x satisfies that property, then every element of W satisfies that property.* This provides a convenient device for proving that a property is satisfied by every element of well ordered set.This feature is called the *theorem* (or *principle*) of *transfinite induction*. Clearly, an initial segment $I(x)$ of an ordered set is the set of al predecessors of x, we can formulate the theorem of transfinite induction as follows:

Theorem (T r a n s f i n i t e I n d u c t i o n) *Let $(W, \leq)$ be a well ordered set and let $V \subseteq W$. Suppose that every element $x \in W$ satisfies the following condition: if every predecessor of x belongs to V, then $x \in V$. Then $V = W$.*