E0 221 Discrete Structures / August-December 2012

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

1-st Midterm : Saturday, September 22, 2012; 14:00 -16:30 **2-nd Midterm :** Sunday, October 14, 2012; 10:00 -12:00
Final Examination : December ??, 2012, 10:00 -13:00

Evaluation Weightage : Midterms (Two) : 50% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

4. Finite Sets – Elementary Counting Techniques

4.1 Let X be a finite set with n elements.

(a) The number of subsets of X is 2^n. (**Hint :** The map $\mathfrak{P}(X) \to \{0,1\}^X$, $A \mapsto e_A$ is bijective.)

(b) If $n \in \mathbb{N}^*$, then the number of subsets of X with an even number of elements is equal to the number of subsets of X with an odd number of elements. Moreover, this number is equal to 2^{n-1}. (**Hint :** Let $a \in X$. The map defined by $A \mapsto A \cup \{a\}$, if $a \notin A$, resp. $A \smallsetminus \{a\}$, if $a \in A$, is a bijective map from the set of subsets with an even number of elements onto the set of subsets with an odd number of elements.)

(c) For $n \in \mathbb{N}$, prove that : $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$. (**Hint :** Use part (a).)

(d) For $n \in \mathbb{N}^*$, prove that : $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0$. (**Hint :** Use part (b) or $(1-1)^n = 0$.)

(e) Prove that $\sum_{k=0}^{n} \binom{2n+1}{2k} = 4^n = \sum_{k=0}^{n} \binom{2n+1}{2k+1}$ for $n \in \mathbb{N}$ and $\sum_{k=0}^{n} \binom{2n}{2k} = \frac{4^n}{2} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}$ for $n \in \mathbb{N}^*$. (**Hint :** Use part (b).)

(f) Let Y be a k-element subset of X. Then the number of m-element subsets of X which contain Y is $\binom{n-k}{m-k}$.

(g) For natural numbers m, n with $m \le n$, show that $\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$. (**Hint :** Compute the sum of all numbers in the part f), where Y runs through all k-element subsets of X in two different ways or use the formula $\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}\binom{m}{k}$.)

(h) For $m, n, k \in \mathbb{N}$, prove that

$$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0}.$$

In particular, $\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$ for $n \in \mathbb{N}$. (**Hint :** Let X, Y be disjoint sets with $|X| = m$, $|Y| = n$. The assignment $A \mapsto (A \cap X, A \cap Y)$ defines a bijective map $\mathfrak{P}(X \cup Y) \to \mathfrak{P}(X) \times \mathfrak{P}(Y)$.)

(i) What is the cardinality of the set $\mathfrak{P}_{\ge n+1}(\{1, 2, \ldots, 2n+1\})$? (see also Test-Exercise T4.3-(b), parts (e) and (d) above.)

4.2 Let X be a finite set with n elements.

(a) Prove that the number of pairs (X_1, X_2) in $\mathfrak{P}(X) \times \mathfrak{P}(X)$ with $X_1 \cap X_2 = \emptyset$ is 3^n. More generally : The number of m–tuples $(X_1, \ldots, X_m)$ of pairwise disjoint subsets $X_1, \ldots, X_m \subseteq X$ is equal $(m+1)^n$.

(b) For $n, r \in \mathbb{N}$, prove that $\sum_m \binom{n}{m} = r^n$, where m run through the set of all r-tuples $(m_1, \ldots, m_r) \in \mathbb{N}^r$ of natural numbers with $m_1 + \cdots + m_r = n$. (**Hint :** Use $r^n = (1 + \cdots + 1)^n$ or the part a).)

4.3 (Sylvester's Sieve--formula [1]) Let $X_1,\ldots,X_n$ be finite sets. For $J\subseteq\{1,\ldots,n\}$, let $X_J:=\bigcap_{i\in J}X_i$ with $X_\emptyset:=\bigcup_{i=1}^n X_i$. Prove that

$$\sum_{J\in\mathfrak{P}(\{1,\ldots,n\})}(-1)^{|J|}|X_J|=0\,,\quad\text{i.e.}\quad |X|=\sum_{\emptyset\neq J\in\mathfrak{P}(\{1,\ldots,n\})}(-1)^{|J|-1}|X_J|\,.$$

(**Hint :** By induction on n. — **Variant :** For $k=1,\ldots,n$, let Y_k be the set of elements $x\in X_\emptyset$ which belong to exactly k of the sets $X_1,\ldots,X_n$. Then $Y_k, 1\le k\le n$ are pairwise disjoint. Using Exercise 4.1 (b) show that

$$\sum_{\substack{J\in\mathfrak{P}(\{1,\ldots,n\})\\ |J|\,\text{even}}}|X_J|=\sum_{k=1}^n 2^{k-1}|Y_k|=\sum_{\substack{J\in\mathfrak{P}(\{1,\ldots,n\})\\ |J|\,\text{odd}}}|X_J|\,.)$$

4.4 Let X be a finite set with m elements.

(a) Let p_m denote the number of permutations of X which donot have fixed points and let $s_m=m!$ be the number of all permutations of X. Show that :

$$\frac{p_m}{s_m}=\frac{1}{0!}-\frac{1}{1!}+\cdots+(-1)^m\cdot\frac{1}{m!}\,.$$

(**Hint :** Let $X=\{x_1,\ldots,x_m\}$. Set $X_i:=\{\sigma\in\mathfrak{S}(X):\sigma(x_i)=x_i\}$ and compute $s_m-p_m=|\bigcup_{i=1}^m X_i|$ using the Sieve formula in Exercise 4.3.) – (**Remark :** Note that $\lim_{m\to\infty}(p_m/s_m)=e^{-1}$, where $e:=\lim_{n\to\infty}(1+\frac{1}{n})^n=2.71828182845904523536\ldots$ is the *Euler's number* which is base of the natural logarithm.) – The number of permutations of X with exactly r fixed points is $\binom{m}{r}p_{m-r}$, $0\le r\le m$. (Proof!)

(b) Let X be a finite set with m elements and let Y be a finite set with n elements. The number of surjective maps from X in Y is

$$n^m-\binom{n}{1}(n-1)^m+\binom{n}{2}(n-2)^m-\cdots+(-1)^n\binom{n}{n}(n-n)^m\,.$$

(**Hint :** Let $Y=\{y_1,\ldots,y_n\}$. Set $P_i:=\{f\in Y^X: y_i\notin \operatorname{im} f\}$ and compute the number $|\bigcup_{i=1}^n P_i|$ of non-surjective maps using the Sieve formula in Exercise 4.3.))

4.5 Let I be a finite index set with n elements and let $\sigma_i\in\mathbb{N}$ for $i\in I$, $\pi:=\prod_{i\in I}\sigma_i$, $\sigma:=\sum_{i\in I}\sigma_i$ and $\sigma_H:=\sum_{i\in H}\sigma_i$ for $H\subseteq I$. Then

$$\sum_{H\subseteq I}(-1)^{|H|}\binom{\sigma_H}{n}=(-1)^n\pi\quad\text{and}\quad\sum_{H\subseteq I}(-1)^{|H|}\binom{\sigma_H}{n+1}=\frac{(-1)^n}{2}(\sigma-n)\pi\,,$$

(**Hint :** Let $X=\bigcup_{i\in I}X_i$, where X_i are pairwise disjoint subsets with $|X_i|=\sigma_i$. For a proof of the first formula consider the set $\mathfrak{P}_n(X)$ and its subsets $Y_i:=\{A\in\mathfrak{P}_n(X)\mid A\cap X_i=\emptyset\}$ and use the Sieve formula in Exercise 4.3 to find $|\bigcup_{i\in I}Y_i|$.)

4.6 Let m,n be two natural numbers.

(a) Let $\mathrm{a}(m,n)$ (respectively, $\mathrm{b}(m,n)$) denote the number of m–tuples $(x_1,\ldots,x_m)\in\mathbb{N}^m$ with $x_1+\cdots+x_m\le n$ (respectively, $x_1+\cdots+x_m=n$). Show that

$$\mathrm{a}(m,n)=\binom{n+m}{m}\ \text{and}\ \mathrm{b}(m,n)=\binom{n+m-1}{m-1}.$$

(**Hint :** Remember to put $\binom{-1}{-1}:=1$. For $m\ge1$, note the equalities $\mathrm{a}(m-1,n)=\mathrm{b}(m,n)$ and $\mathrm{a}(m,n)=\mathrm{a}(m,n-1)+\mathrm{a}(m-1,n)$ and then use induction on $n+m$. – **Variant :** The map $(x_1,\ldots,x_m)\mapsto\{x_1+1,x_1+$

[1] This formula is attributed to Joseph Sylvester. James Joseph Sylvester (1814-1897) was an English mathematician. He made fundamental contributions to matrix theory, invariant theory, number theory, partition theory and combinatorics. He played a leadership role in American mathematics in the later half of the 19th century as a professor at the Johns Hopkins University and as founder of the American Journal of Mathematics. It is sometimes also named for Abraham de Moivre, Daniel da Silva or Henri Poincaré.

$x_2+2,\ldots,x_1+\cdots+x_m+m\}$ maps the set of m–tuples $(x_1,\ldots,x_m)\in\mathbb{N}^m$ with $x_1+\cdots+x_m\le n$ bijectively onto the set of m–element subsets of $\{1,2,\ldots,n+m\}$.)

(b) Suppose that $m\ge 1$. Prove that the number of m–tuples $(x_1,\ldots,x_m)\in(\mathbb{N}^+)^m$ of positive natural numbers with $x_1+\cdots+x_m=n$ is $\binom{n-1}{m-1}$.

(c) Let $k\in\mathbb{N}$ with $k\le n$. Prove that the subset

$$\mathfrak{X}=\{A\in\mathfrak{P}_k(\{1,\ldots,n\})\mid \text{if } a\in A,\ \text{ then } a+1\notin A\}$$

of $\mathfrak{P}_k(\{1,\ldots,n\})$ has cardinality $\binom{n-k+1}{k}$.

(d) Let $X=\{x_1,\ldots,x_{2n+1}\}$, $n\in\mathbb{N}$ be a set with $2n+1$ elements. For $k=0,1,\ldots,n$, let $\mathfrak{X}_k$ be the set of all those subsets of X of cardinality $\ge n+1$ which contain x_{n+k+1} and exactly n elements from $x_1,\ldots,x_{n+k}$, i.e.

$$\mathfrak{X}_k=\{A\in\mathfrak{P}_{\ge n+1}(X)\mid |A\cap\{x_1,\ldots x_{n+k}\}|=n \text{ and } x_{n+k+1}\in A\}.$$

Show that $\bigcup_{k=0}^n\mathfrak{X}_k=\mathfrak{P}_{\ge n+1}(X)$ and hence deduce that $\sum_{k=0}^n 2^{n-k}\binom{n+k}{k}=4^n$. Note that subsets of X which are elements of $\mathfrak{X}_k$ may contain some elements from $x_{n+k+2},\ldots,x_{n+1}$. See also Test-Exercise T4.3-(b), 4.1-(e) and 4.1-(i).)

4.7 Let $X_1,\ldots,X_n$ be finite subsets of a finite set Ω. For $\emptyset\ne J\subseteq\{1,\ldots,n\}$, let $X_J:=\bigcap_{i\in J}X_i$ and $X:=X_\emptyset:=\bigcup_{i=1}^n X_i$. Further, for $j=1,\ldots,n$, put $\xi_j:=\sum_{J\in\mathfrak{P}_j(\{1,\ldots,n\})}|X_J|$ and $\xi_0:=|\Omega|$. Prove that

(a) $\left|\bigcap_{i=1}^n(\Omega\setminus X_i)\right|=\sum_{j=0}^n(-1)^j\xi_j$. (**Hint :** By Sylvester's sieve formula (Exercise 4.3), $|X|=\sum_{j=1}^n(-1)^{j-1}\xi_j$. Since $\cap_{i=1}^n(\Omega\setminus X_i)=\Omega\setminus\cup_{i=1}^n X_i=\Omega\setminus X$, we get $|\cap_{i=1}^n(\Omega\setminus X_i)|=|\Omega|-|X|$.)

(b) For $k=1,\ldots,n$, let Y_k be the set of all those elements in X which belongs to exactly k of the subsets $X_1,\ldots,X_n$. Then show that $|Y_m|=\sum_{r=m}^n(-1)^{r-m}\binom{r}{m}\xi_r$ for all $1\le m\le n$. (**Hint :** Let $1\le k,m\le n$ and let m be fixed. Suppose that $x\in Y_k$ and (may) assume that $x\in X_1,\ldots,X_k$ and $x\notin X_i$ for all $k<i\le n$. If $k<m$, then $x\notin Y_m$ and hence x does not contribute anything to ξ_r for $r\ge m$. If $k=m$, then $x\in Y_m$ and in the sum on the LHS it contributes only to one term, namely, to $\binom{m}{m}\xi_m$, since $\xi_m:=\sum_{J\in\mathfrak{P}_m(\{1,\ldots,n\})}|X_J|$ and only one of these intersections, namely, $X_1\cap\cdots\cap X_m$ contains x. In the remaining case $k>m$, $x\notin Y_m$ and hence x contributes nothing. On the other hand its contribution to ξ_r is $\binom{k}{r}$ (one in each $J\in\mathfrak{P}_r(\{1,\ldots,k\})$. Therefore if we let $j=r-m$, then the problem redusces to prove the identity $\sum_{j=0}^{k-m}(-1)^j\binom{m+j}{m}\binom{k}{m+j}=0$ which is stated in Test-Exercise T4.4-(b)-(2).)

4.8 The purpose of this Exercise is to give an alternative proof of the Exersicse 4.7-(b). Let Ω be a finite set and let $f:\Omega\times\mathfrak{P}(\Omega)\to\mathbb{R}$ be the map defined by $(x,A)\mapsto\begin{cases}0, & \text{if } x\notin A,\\ 1, & \text{if } x\in A.\end{cases}$

Show that :

(a) For each $A\in\mathfrak{P}(\Omega)$, the map $f(-,A)$ is the indicator function e_A of A. In particular, for any two subsets $A,B\in\mathfrak{P}(\Omega)$, we have :

(1) $f(x,\Omega\setminus A)=1-f(x,A)$;
(2) $f(x,A\cap B)=f(x,A)\cdot f(x,B)$;
(3) $f(x,A\cup B)=f(x,A)+f(x,B)-f(x,A\cap B)$;
(4) $|A|=\sum_{x\in\Omega}f(x,A)$. (**Hint :** See the Exercise 1.??.)

(b) Let $I:=\{1,2,\ldots,n\}$ and let $X_1,\ldots,X_n\in\mathfrak{P}(\Omega)$ and for each $J\in\mathfrak{P}(I)$, let $X_J:=\cap_{j\in J}X_j$ (and $X_\emptyset:=\Omega$). Then prove that $\sum_{J\in\mathfrak{P}_j(I)}|X_J|=\sum_{x\in\Omega}\big(\sum_{J\in\mathfrak{P}_j(I)}f(x,X_J)\big)$. (**Hint :** Use the part (a).)

(c) If an element $x \in \Omega$ belongs to exactly k of the subsets $X_1, \ldots, X_n$, then prove that

$$\sum_{J \in \mathfrak{P}_r(I)} f(x, X_J) = \binom{k}{r}.$$

(**Hint :** Here we use the understanding that $\binom{0}{0} = 1$. We may assume that $x \in X_1 \cap \cdots \cap X_k$ and $x \notin X_i$ for all $k < i \leq n$. For every $J \in \mathfrak{P}_r(\{1, \ldots, n\})$, $f(x, X_J) = \prod_{j \in J} f(x, X_j) = 1$ if and only if $J \subseteq \{1, \ldots, k\}$, i.e., $J \in \mathfrak{P}_r(\{1, \ldots, k\})$. This proves that LHS is equal to the cardinality $|\mathfrak{P}_r(\{1, \ldots, k\})| = \binom{k}{r}$.)

(d) For every $x \in \Omega$, show that $f(x, \cap_{i=1}^n (\Omega \setminus X_i)) = \sum_{j=0}^{n} (-1)^j \Big(\sum_{J \in \mathfrak{P}_j(I)} f(x, X_J) \Big)$. (**Hint :** For $i \in I := \{1, \ldots, n\}$, put $X'_i := \Omega \setminus X_i$. Then by (a)-(1) , (2) LHS $= \prod_{i=1}^n f(x, X'_i) = \prod_{i=1}^n (1 - f(x, X_i)) = 1 + \sum_{j=1}^n (-1)^j \sum_{J \in \mathfrak{P}_j(I)} (\prod_{k \in J} f(x, X_k)) = 1 + \sum_{j=1}^n (-1)^j \sum_{J \in \mathfrak{P}_j(I)} f(x, X_J))$. – **Remark :** Suming over the two sides of this formula as x varies over Ω and using the parts (a) and (b), we get the proof of the formula given in the Exercise 4.7-(b).)

4.9 For $k \in \mathbb{N}^+$, a k-ary sequence is a sequence with values in a finite set with k elements (generally in the set $\{0, \ldots, k-1\}$), i.e. a k-ary sequence is an element in the set $\{0, \ldots, k-1\}^{\mathbb{N}}$. For $k = 2, 3, 4, 5$ these sequences are also called binary, ternary, quaternary, quintnary sequences. (See also Test-Exercise T4.2-(c).)

(a) Show that the number of binary sequences of length n in which the digit 1 occurs even number of times is 2^{n-1}. This is also the number of binary sequences of length n in which the digit 1 occurs odd number of times. (**Hint :** Let $X := \{0,1\}^{\{1,\ldots,n\}}$ be the set of all binary sequences of length n and let $X_{\text{even}}(1)$ (respectively, $X_{\text{odd}}(1)$ be the set of all binary sequences of length n in which the digit 1 occurs even (respectively, odd) number of times. Then clearly $X = X_{\text{even}}(1) \uplus X_{\text{odd}}(1)$. First assume that n is odd. Then the map $f : X \to X$ defined by $f((a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$, where $a'_i = 0$ or 1 according as $a_i = 1$ or 0 for all $i = 1, \ldots n$, is a bijection. Moreover, if n is odd, then $f(X_{\text{even}}(1)) = X_{\text{odd}}(1)$ and $f(X_{\text{odd}}(1)) = X_{\text{even}}(1)$. Therefore $|X_{\text{even}}(1)| = |X_{\text{odd}}(1)|$ and $2^n = |X| = |X_{\text{even}}(1)| + |X_{\text{odd}}(1)| = 2 \cdot |X_{\text{even}}(1)| = 2 \cdot |X_{\text{odd}}(1)|$. Now, if n is even, then one can reduce the computation to the case when n is odd : Let $A := \{(a_1, \ldots, a_n, a_{n+1}) \in X \mid a_{n+1} = 1$ and $B := \{(a_1, \ldots, a_n, a_{n+1}) \in X \mid a_{n+1} = 0$. Then $|A| = |B| = 2^{n-1}$ and hence $X = A \uplus B$. Further, $X_{\text{even}}(1) = (A \cap X_{\text{even}}(1)) \uplus (B \cap X_{\text{even}}(1))$ and hence $|X_{\text{even}}(1)| = |(A \cap X_{\text{even}}(1))| + |(B \cap X_{\text{even}}(1))| = 2^{n-2} + 2^{n-2} = 2^{n-1}$, since $n-1$ is odd. Finally, $|X_{\text{odd}}(1)| = |X| - |X_{\text{even}}(1))| = 2^n - 2^{n-1} = 2^{n-1}$.)

(b) Show that the number of k-ary sequences of length n in which the digit 1 occurs even number of times is $\dfrac{k^n + (k-2)^n}{2}$. (**Hint :** Let $Y := \{2, 3, \ldots, k-1\}^{\{1,\ldots,n\}}$ denote the set of all those k-ary sequences of length n which do not contain 0 or 1 and let $Z : X \setminus Y$. Classify the sequences in Z by their pattern, i.e., consider the equivalence classes $\sim$ $Z_1, \ldots, Z_s$ with respect to the equivalence relation on Z. Then $|Z| = |Z_1| + \cdots + |Z_s|$. Note that by definition Z_i is the set of all k-ary sequences of length n which have the same pattern of the symbols $2, 3, \ldots, k-1$ and hence $|Z_i| = 2^{n-r}$, where r is the number of places filled by the symbols $2, 3, \ldots, k-1$. Now by part a) half of these sequences have even number of 1's and this is true for all $i = 1, \ldots, s$. This proves that $|Z_{\text{even}}(1)| = \sum_{i=1}^s \frac{1}{2}|Z_i| = \frac{1}{2}|Z| = \frac{1}{2}(k^n - (k-2)^n)$. Therefore, since $X_{\text{even}}(1) = Y \uplus Z_{\text{even}}(1)$, we get $|X_{\text{even}}| = |Y| + |Z_{\text{even}}(1)| = (k-2)^n + \frac{1}{2}(k^n - (k-2)^n)$.)

(c) For positive natural numbers $n, k \in \mathbb{N}^+$, $k \geq 2$, prove the formula :

$$\sum_{r \in \mathbb{N}} \binom{n}{2r} (k-1)^{n-2r} = \frac{k^n + (k-2)^n}{2}.$$

(**Hint :** Follows from the part b), since the sum on the left is the number of k-ary sequences of length n in which the digit 1 occurs even number of times.)

(d) Show that the number of k-ary sequences of length n in which both 0 and 1 occur even number of times is $\dfrac{k^n + 2(k-2)^n + (k-4)^n}{4}$, $k \geq 2$. (**Hint :** Let 1 occur $2r$ times in a k-ary sequences of length n. Then the remaining $(k-1)$-ary sequence is of length $n-2r$. If 0 occur in an even number of times, then by part (b), there are $\frac{(k-1)^{n-2r}+(k-3)^{n-2r}}{2}$ such sequences. Now the assertion follows by applying the part (c) twice (once for k and then for $k-2$) and adding.)

(e) Find the number of k-ary sequences of length n in which the digit 1 occurs even number of times and the digit 0 occurs odd number of times. (**Hint :** The answer is $\frac{k^n-(k-4)^n}{4}$ – From the k-ary sequences of length n in which the digit 1 occur even number of times, remove the k-ary sequences of length n in which the digit 0 occur even number of times, i.e. compute

$$\sum_{r\in\mathbb{N}}\binom{n}{2r}\big[(k-1)^{n-2r}-\frac{(k-1)^{n-2r}+(k-3)^{n-2r}}{2}\big].)$$

4.10 Prove the following (marriage[2]) theorem : *Let Y_x, $x\in X$, be a finite family of sets. For every subset N of X assume that the set $Y_N := \cup_{x\in N} Y_x$ has at least $|N|$ elements, i. e., $|Y_N|\geq|N|$ for every $N\in\mathfrak{P}(X)$. Then there exists an injective map $f: X\to Y_X$ with $f(x)\in Y_x$ for every $x\in X$.*

(**Proof :** Proof by induction on $n=|X|$. The case of n = 1 and a single pair liking each other requires a mere technicality to arrange a match. For the inductive step consider two cases :

Case 1: $|Y_N|>|N|$ for every subset $N\subseteq X$, $N\neq\emptyset$, $N\neq X$. In this case for $x\in X$, choose $y\in Y_x$ and consider $X' := X\setminus\{x\}$ and $Y'_{x'} := Y_x\setminus\{y\}$, $x'\in X$. Then clearly the marriage condition still holds and hence by the inductive hypothesis, there is an injective map $f': X'\to Y'_{X'}$ with $f'(x')\in Y'_{x'}$. Now, define $f: X\to Y_X$ by $f(x)=y$ and $f(x')=f'(x')$.

Case 2: There exists a subset $\emptyset\neq N\subset X$, with $|Y_N|=|N|$. In this case, by the inductive hypothesis, there exists an injective (in fact bijective) map $g: N\to Y_N$. The trick is to show that $X'' := X\setminus N$ and $Y''_{x''} := Y_{x''}\setminus Y_N$, $x''\in X''$ satisfy the marriage condition, then by the inductive hypothesis, there is an injective map $X''\to Y''_{X''}$ with $f''(x'')\in Y''_{x''}$. Now, define $f: X\to Y_X$ by $f(x)=g(x)$ for $x\in N$ and $f(x'')=f''(x'')$ for $x''\in X''$. •)

(a) Let $\mathfrak{P}=(X_1,\ldots,X_r)$ and let $\mathfrak{q}=(Y_1,\ldots,Y_r)$ be partitions of the set X into r pairwise disjoint subsets each of them with $n\geq 1$ elements. Show that $\mathfrak{P}$ and $\mathfrak{q}$ has a common representative system, i.e. there exist r distinct elements $x_1,\ldots,x_r$ in X such that each x_i belongs to exactly one of the subset $X_1,\ldots,X_r$ and exactly one of the subset $Y_1,\ldots,Y_r$.

(**Hint :** Using the above Marriage-theorem find a permutation $\sigma\in\mathfrak{S}_r$ such that $X_i\cap Y_{\sigma(i)}\neq\emptyset$ for every $1\leq i\leq r$. – **Remark :** The assumption that $|X_i|=|Y_i|=n$ for all $i=1,\ldots,r$ can be replaced by some what weaker condition : for every subset $J\subseteq\{1,\ldots,r\}$, the subset $X_J := \cup_{j\in J} X_j$ contains at most $|J|$ components $Y_1,\ldots,Y_r$ of $\mathfrak{q}$.)

(b) Let $\mathfrak{A}$ be the $n\times r$ integral matrix

$$\mathfrak{A}=\begin{pmatrix}1 & 2 & \cdots & r\\ r+1 & r+2 & \cdots & 2r\\ \vdots & \vdots & \ddots & \vdots\\ (n-1)r+1 & (n-1)r+2 & \cdots & nr\end{pmatrix}$$

[2] This theorem is popularly known as the (m a r r i a g e - t h e o r e m) and it provides the solution for the m a r r i a g e p r o b l e m which requires to match n girls with the set of n boys. Each girl (after a long and no doubt exhausting deliberation) submits a list of boys she likes. We also make an assumption that being of noble character no boy will break a heart of a girl who likes him by turning her down. Sometimes all the girls can be given away, sometimes no complete match is possible. Therefore for a complete match a (marriage) condition is necessary; the marriage condition can be formulated in several equivalent ways, for example, *For each $r=1,\ldots,n$ every set of r girls likes at least r boys.* (or equivalently, *For each $r=1,\ldots,n$ every set of r boys likes at least r girls.)* The marriage condition and the marriage theorem are due to the English mathematician P h i l i p H a l l (1904-1982). Hall was the main impetus behind the British school of group theory and the growth of group theory to be one of the major mathematical topics of the 20th Century was largely due to him. Therefore Marriage theorem is precisely: *Hall's marriage condition is both sufficient and necessary for a complete match.* The necessary part is obvious. The sufficient part is shown by induction on $n=|X|$.

and let $\mathfrak{B}$ be another $n\times r$ integeral with entries $1,2,\ldots,nr$ (at arbitrary positions). Show that there exists a permutation $\sigma\in\mathfrak{S}_r$ such that for every $i=1,\ldots,r$, the i-th column of $\mathfrak{A}$ and the $\sigma(i)$-th column of $\mathfrak{B}$ contain at least one element in common. (**Hint :** Use the part (a).)

(c) Let G be a finite group and let H be a subgroup of G. Let $G=Hy_1\cup\cdots\cup Hy_r$ (respectively, $G=z_1H\cup\cdots\cup z_rH$) be a right-coset (respectively, left-coset) decomposition for G. Show that there exist elements $x_1,\ldots,x_r\in G$ such that $G=Hx_1\cup\cdots\cup Hx_r=x_1H\cup\cdots\cup x_rH$. (**Hint :** Use the part (a).)

(d) Let X be a finite set with n elements. For $i\in\mathbb{N}$, let $\mathfrak{P}_i(X)$ be the set of all subsets Y of X with $|Y|=i$. Show that: If $i\in\mathbb{N}$ with $0\le i<n/2$ (respectively, with $n/2<i\le n$), then there exists an injective map $f_i:\mathfrak{P}_i(X)\to\mathfrak{P}_{i+1}(X)$ such that $Y\subseteq f_i(Y)$ for all $Y\in\mathfrak{P}_i(X)$ (respectively, an injective map $g_i:\mathfrak{P}_i(X)\to\mathfrak{P}_{i-1}(X)$ such that $g_i(Y)\subseteq Y$ for all $Y\in\mathfrak{P}_i(X)$). (**Hint :** Let $0\le i<n/2$. A pair $(Y,Y')\in\mathfrak{P}_i(X)\times\mathfrak{P}_{i+1}(X)$ is called *amicable* if $Y\subseteq Y'$. Let $\mathfrak{R}$ be a subset of $\mathfrak{P}_i(X)$ with $|\mathfrak{R}|=:r$. Further, let $\mathfrak{R}'$ be the set of all those $Y'\in\mathfrak{P}_{i+1}(X)$ which are amicable to at least one $Y\in\mathfrak{R}$. Put $s:=|\mathfrak{R}'|$. Then $r(n-i)\le s(i+1)$ and hence $r\le s$. Now use the marriage-theorem.)

4.11 Let X be a finite set with n elements.

(a) Let $(m_1,\ldots,m_r)\in\mathbb{N}^r$ be such that $m_1+\cdots+m_r=n$. Show that the number of partitions $\mathfrak{p}=(X_1,\ldots,X_r)$ of X with $|X_i|=m_i$, for all $i=1,\ldots,r$, is the polynomial coefficient

$$\binom{n}{m}:=\frac{n!}{m!}=\frac{n!}{m_1!\cdots m_r!}\,.$$

(**Hint :** Fix a partition $\mathfrak{q}=(Y_1,\ldots,,Y_r)$ of X with $|Y_i|=m_i$, $i=1,\ldots,r$ and define a map $\mathfrak{S}(X)\longrightarrow\mathfrak{Z}:=\{\mathfrak{p}=(X_1,\ldots,X_r)\in\mathfrak{Par}_r(X)\mid |X_i|=m_i, i=1,\ldots,r\}$, be the map defined by $f\mapsto\mathfrak{p}(f):=(f(X_1),\ldots,f(X_r))$. Show that all the fibres of this map have the same cardinality $=m!=m_1!\cdots m_r!$. Now use the Shepherd-rule.)

(b) (Stirling numbers of second kind[3]) For $n,r\in\mathbb{N}$ with $0\le r\le n$, put $\mathrm{S}(n,r):=|\mathfrak{Par}_r(X)|$, where $\mathfrak{Par}_r(X)$ is the set of all partitions $\mathfrak{p}=(X_1,\ldots,X_r)$ of X into r subsets. For all other pairs $(n,r)\in\mathbb{Z}^2$, we put $\mathrm{S}(n,r)=0$.

Show that

(1) For $n\ge1$, $\mathrm{S}(n,2)=2^{n-1}-1$.

(2) $\mathrm{S}(n,r)=\frac{1}{r!}|\mathrm{Maps_{surj}}(X,\{1,\ldots,r\})|=\frac{1}{r!}\sum_{k=0}^{r}(-1)^k\binom{r}{k}(r-k)^n=\sum_{k=0}^{r}\frac{(-1)^k(r-k)^{n-1}}{k!\cdot(r-k-1)!}\,.$

In particular, $r!=\sum_{i=0}^{r}(-1)^k\binom{r}{k}(r-k)^r$.

(3) $\sum_{k=1}^{n}k!\cdot\binom{r}{k}\cdot\mathrm{S}(n,k)=r^n$.

(**Hint :** To prove (1) show that each fibre of the map $\mathfrak{P}(X)\setminus\{\emptyset,X\}\to\mathfrak{Par}_2(X)$ defined by $Y\mapsto(Y,X\setminus Y)$ has cardinality 2 and hence $2^n-2=|\mathfrak{P}(X)\setminus\{\emptyset,X\}|=2\cdot|\mathfrak{Par}_2(X)|$ by Shepherd-rule. To prove (2) show that each fibre of the map $\mathrm{Maps_{surj}}(X,\{1,\ldots,r\})\to\mathfrak{Par}_r(X)$ defined by $f\mapsto(f^{-1}(1),\ldots,f^{-1}(r))$ has cardinality $r!$ and hence by the Shepherd-rule and Exercise 4.4-(b), we have $r!\cdot|\mathfrak{Par}_r(X)|=|\mathrm{Maps_{surj}}(X,\{1,\ldots,r\})|$. The last part follows from the equality $\pi(r,r)=1$. For the proof of (3), compute the cardinality of each fibre of the map

$$\mathrm{Maps}(X,\{1,\ldots,r\})\to\biguplus_{k=1}^{n}\mathfrak{P}_k(\{1,\ldots,r\})\times\mathfrak{Par}_k(X))\,,f\mapsto(f(X),\mathfrak{p}(f))\,,$$

where $\mathfrak{p}(f):=(f^{-1}(i))_{i\in f(X)}$ and then use (2). – **Remarks :** The Stirling numbers appear in many other problems. Clearly $\mathrm{S}(n,r)=0$ for $r>n$, $\mathrm{S}(n,n)=1$, $\mathrm{S}(n,1)=1$; $\mathrm{S}(n,n-1)=\binom{n}{2}$; a less trivial result is

[3] James Stirling (1692-1770) was a Scottish mathematician whose most important work Methodus Differentialis in 1730 is a treatise on infinite series, summation, interpolation and quadrature.

the formula for $\mathrm{S}(n,2)$ given in the part (1). For $r>2$, there is no easy formula for $\mathrm{S}(n,r)$. For small values of n and r one can find $\mathrm{S}(n,r)$ by actually considering all partitions of a set with n elements. For higher values this becomes impracticable and also unreliable. The important recurrence relation given below in c) which allows us to compute a Stirling numbers by first computing the lower Stirling numbers. Consider the polynomial $F(T) := T^n - \sum_{k=0}^{n} k!\cdot \mathrm{S}(n,k)\cdot \binom{T}{k}$, where $\binom{T}{k} := \frac{T(T-1)\cdots(T-k+1)}{k!}$ are the binomial polynomials of degree k. Then, since $F(T)$ is a polynomial of degree $\le n$ with integer coefficients and by (3), the integers $0,1,\ldots,n$ are $n+1$ distinct zeroes of F, we conclude that $F=0$ and therefore the Stirling numbers od second kind are also defined by the polynomial equation $T^n = \sum_{k=0}^{n} k!\cdot\mathrm{S}(n,k)\cdot\binom{T}{k}$. If one takes this as the definition of the Stirlings numbers $\mathrm{S}(n,r)$ of second kind, then (1) and (3) are immediate by putting $T=2$ and $T=r$ respectively. This also leads to the definition of the Stirling numbers of first kind : For $r,n\in\mathbb{N}$ with $0\le r\le n$, let $\mathrm{S}(n,r)\in\mathbb{Z}$ be defined by the polynomial equation : $\binom{T}{n} = \frac{1}{n!}\cdot\sum_{r=0}^{n}(-1)^{n-r}\cdot\mathrm{S}(n,r)\cdot T^r$. Put $\mathrm{S}(n,r)=0$ otherwise. For the existence of the numbers $\mathrm{S}(n,r)$ use the fact that $1,T,\ldots,T^n$ and $\binom{T}{0},\binom{T}{1},\ldots,\binom{T}{n}$ are two bases of the $\mathbb{Q}$-vector space $\mathbb{Q}[T]_n$ of polynomials with rational coefficients of degree $\le n$.)

(c) The Stirling numbers of second kind satisfy the recursion relations :

$$\mathrm{S}(0,r)=\delta_{0r}\,,\quad \text{and}\quad \mathrm{S}(n+1,r) = r\mathrm{S}(n,r)+\mathrm{S}(n,r-1)\,,$$

where δ_{ij} denote the *Kronecker's delta*. (**Hint :** From $\binom{T}{k+1} = T\cdot\binom{T}{k} - k\cdot\binom{T}{k}$, we get $T^{n+1} = \sum_{k=0}^{n} k!\cdot \mathrm{S}(n,k)\cdot T\cdot\binom{T}{k} = \sum_{k=0}^{n+1} k!\cdot[k\cdot\mathrm{S}(n,k)+\mathrm{S}(n,k-1)]\cdot\binom{T}{k}$. — **Remark :** The Stirling numbers of first kind satisfy the recursion relations : $\mathrm{S}(0,r)=\delta_{0r}$, and $\mathrm{S}(n+1,r) = n\cdot\mathrm{S}(n,r)+\mathrm{S}(n,r-1)$.))

(d) Prove that $\beta_n = \sum_{r=0}^{n}\mathrm{S}(n,r)$ for every $n\in\mathbb{N}$. (**Hint :** See Test-Exercise T4.?-(?) and use Test-Exercise T4.?-(?).))

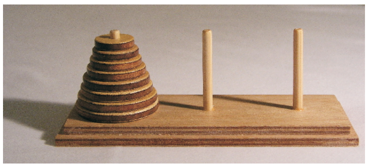
(e) Prove that $\mathrm{S}(n+1,r) = \sum_{k=1}^{n}\binom{n}{k}\mathrm{S}(k,r-1) = \sum_{k=0}^{n} r^{n-k}\mathrm{S}(k,r-1)$.

(**Hint:** The second equality is proved by induction and using recursion relations (see part (c)) :
$\mathrm{S}(n+1,r) = r\,\mathrm{S}(n,r)+\mathrm{S}(n,r-1) = \sum_{k=r-1}^{n-1} r\cdot r^{n-1-k}\mathrm{S}(k,r-1)+\mathrm{S}(n,r-1) = \sum_{k=r-1}^{n} r^n\,\mathrm{S}(k,r-1) = \sum_{k=0}^{n} r^n\,\mathrm{S}(k,r-1)$.
For the first equality consider the map $\biguplus_{k=0}^{k}\big(\biguplus_{I\in\mathfrak{P}_k(X)}\{I\}\times\mathfrak{Par}_{r-1}(I)\big)\longrightarrow\mathfrak{Par}_k(X\uplus\{y\})$ defined by $(I,(I_1,\ldots,I_{r-1}))\mapsto((X\setminus I)\uplus\{y\},I_1,\ldots,I_{r-1})$.)

Below one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

T4.1 (Tower of Hanoi) The puzzle consists of n disks of decreasing diameter placed on a pole. There are two other poles. The problem is to move the entire pile to another pole by moving one disk at a time to any other pole, except that no disk may be placed on top of a smaller disk.



Find a formula for the least number of moves needed to move n disks from one pole to another, and prove the formula by induction.

T4.2 (Indicator functions) Let I be a set. For a subset $J \in \mathfrak{P}(I)$, let $e_J : I \to \{0,1\}$ be the indicator function of J (with respect to I), i.e. $e_J(i) = \begin{cases} 1, & \text{if } i \in J, \\ 0, & \text{if } i \in I \setminus J. \end{cases}$. Note that $e_I = 1$ and $e_\emptyset = 0$. Show that

(a) The map $J \mapsto e_J$ is a bijective map from the power set $\mathfrak{P}(I)$ onto the set $\{0,1\}^I$ of all maps $I \to \{0,1\}$.

(b) For subsets $J, K \subseteq I$, prove that : $e_{J\cap K} = e_J e_K\,,\quad e_{J\cup K} = e_J + e_K - e_J e_K\,,\quad e_{J\setminus K} = e_J(1-e_K)\,.$ In particular, $e_{I\setminus J} = 1 - e_J$ and $e_{J\triangle K} = e_J + e_K - 2e_J e_K\,.$

(c) For $J, K \in \mathfrak{P}(I)$, let $J + K := J\Delta K := (J\cup K)\setminus(J\cap K)$ denote the symmetric difference of J and K. Show that :

(1) $J+K = K+J$ and $J+\emptyset = J$, $J+J=\emptyset$.

(2) $(J+K)+L = J+(K+L)$ for all $J,K,L \in \mathfrak{P}(I)$.

(3) For every $J, L \in \mathfrak{P}(I)$, there exists a unique K such that $J+K = L$.

(4) $(J+K)\cap L = (J\cap L)+(K\cap L)$ for all $J,K,L \in \mathfrak{P}(I)$.

(**Remark :** For verification of these properties use indicator functions and their rules given in (b). These properties of the symmetric difference $\triangle$ show that the power set $\mathfrak{P}(I)$ with the symmetric difference Δ as addition and the intersection $\cap$ as multiplication is a commutative ring with $\emptyset$ as the zero element 0 and I as the unit element 1. This ring $(\mathfrak{P}(I), \triangle, \cap)$ is called the set-ring of I. Moreover, it is a finite dimensional algebra over the prime field $\mathbb{Z}_2$ of dimension $|I|$; if $|I|=1$, then it is the prime field $\mathbb{Z}_2$; in the case $|I|>1$, it is not a field – nor even an integral domain.)

T4.3 Use induction to prove that : For all $n \in \mathbb{N}$:

(a) $\sum_{k=0}^{n} k\cdot(k!) = (n+1)!-1$. (b) $\sum_{k=0}^{n} 2^{n-k}\binom{n+k}{k} = 4^n$. (see also Exercise 4.6-(d).)

(c) $\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, $m \in \mathbb{N}$, $m \le n$.

T4.4 **(a)** Let X, Y be finite sets and $Z := X \times Y$. For $x \in X$, let $P_x := \{y \in Y \mid (x,y) \in Z\}$ and for $y \in Y$, let $Q_y := \{x \in X \mid (x,y) \in Z\}$. Then show that $\sum_{x\in X}|P_x| = \sum_{y\in Y}|Q_y|$.

(b) Let $r,k,n,m \in \mathbb{N}$.

(1) If $r \le k \le n$, then $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$. (**Hint :** Just compute both sides!. **Variant :** Suppose from n objects we choose k and put a white tag on the selected objects. Then out of these k objects we select r objects and put a black tag on those selected. This is equivalent to selecting r objects (and putting white

and a black tag on each) and then selecting $k-r$ objects from the remaining $n-r$ putting a white tag on the the selected objects.)

(2) If $m \le k$, then $\sum_{j=0}^{k-m}(-1)^j\binom{m+j}{m}\binom{k}{m+j} = 0$.

(**Hint :** $\sum_{j=0}^{k-m}(-1)^j\binom{m+j}{m}\binom{k}{m+j} = \sum_{j=0}^{k-m}(-1)^j\binom{k}{m}\binom{k-m}{j} = \binom{k}{m}\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j} = 0$ by Exercise 4.1-(d).)

(c) Let $k \in \mathbb{N}^+$ be a positive natural number.

(1) Find how many *palindromes*[4] of length n can be formed with an alphabet of k letters. (**Ans :** k^m if $n = 2m$ and k^{m+1} if $n = 2m+1$.)

(2) How many k-ary sequences of length n are there? (**Ans :** $k^n = |\{0,1,\ldots,k-1\}^{\{1,\ldots,n\}}|$.)

(3) How many k-ary sequences of length n are there in which no two consecutive entries are the same? (**Ans :** $k(k-1)^{n-1}$.)

(4) How many ternary sequences of length n are there which either start with 012 or end with 012? (**Ans :** 0 if $n \le 2$; $2 \cdot 3^{m-3}$, if $3 \le n \le 5$; and $2 \cdot 3^{n-3} - 3^{n-6}$, if $n \ge 6$.)

T4.5 (Relations) Let X and Y be sets. A (binary) relation [5] R from X and Y is a subset $R \subseteq X \times Y$, i.e. an element $R \in \mathfrak{P}(X \times Y)$. For the expression "$(x,y) \in R$" we shall write "$xRy$" and say that "*$x$ is related to y with respect to R*", $x \in X$, $y \in Y$. The set of relations $\mathfrak{P}(X \times Y)$ from X to Y is also denoted by $\operatorname{Rel}(X,Y)$ and its elements are also denoted by the symbols $\sim$, $\cong$ $\equiv$, $\le$, $\preceq$ $\cdots$. In the case $Y = X$, we put $\operatorname{Rel}(X) = \operatorname{Rel}(X,X) = \mathfrak{P}(X,X)$ and its elements are called relation on X.

(a) The map $\Gamma : \operatorname{Maps}(X,Y) \to \mathfrak{P}(X \times Y)$ defined by $f \mapsto \Gamma_f := \{(x, f(x)) \mid x \in X\}$ the graph of f is injective. (**Remark :** Therefore (if we identify maps with its graphs) every map from X to Y is a relation from X to Y. Further, since the map Γ is not surjective if $X \neq \emptyset$ and $(|X|,|Y|) \neq (1,1)$, in this case there are relations from X to Y which are not maps from X to Y. For example, each of the relations $\{(x,y),(x,y') \mid x \in X \ ; \ y,y' \in Y, y \neq y'\}$ and (if $|X| > 1$) $\{(x,y) \mid x \in X, y \in Y\}$ from X to Y is not a map from X to Y. The graph of the identity map $\mathrm{id}_X : X \to X$ is the diagonal $\Delta_X := \{(x,x) \mid x \in X\}$ and hence the diagonal relation Δ_X from X to X is also called the identity relation on X. The relation $R = \emptyset$ and $R = X \times Y$ are called the empty-relation and the all-relation from X to Y, respectively. Further, we can also define intersection and union of arbitrary family of relations.)

(b) The map $\mathfrak{P}(X \times Y) \to \mathfrak{P}(Y)^X$ defined by $R \mapsto \big(x \mapsto \{y \in Y \mid xRy\}\big)$ is bijective. What is the inverse of this map? (**Remark :** With this bijection, one can identify every relation $R \subseteq X \times Y$ between X and Y as a map from X into $\mathfrak{P}(Y)$.)

T4.6 Let X be a set. A relation $R \in \mathfrak{P}(X \times X)$ on X is called (1) reflexive if xRx for all $x \in X$; (2) symmetric if for $x,y \in X$, xRy implies yRx; (3) transitive if for $x,y,z \in X$, xRy and yRz implies xRz; (4) anti-symmetric if for $x,y \in X$, xRy and yRx implies $x = y$.

(a) (Equivalence relations) A relation R on X is called an equivalence relation if it is reflexive, symmetric and transitive. The identity relation δ_X and the all-relation $X \times X$ on X are clearly equivalence relations on X.

Let R be an equivalence relation on X. Then for $x \in X$, the subset $[x]_R = [x] = \{a \in X \mid (a,x) \in R\}$ is called the equivalence class of x under R (sometimes equivalence classes are also denoted by $\overline{a}$).

(1) For every $x \in X$, $x \in [x]$. In particular, $[x] \neq \emptyset$ for every $x \in X$ and $X = \bigcup_{x \in X}[x]$.

(2) For all $x,y \in X$, the following statements are equivalent :

[4] A palindrome is a word which reads the same backward or forward, e. g., "MADAM", "ANNA".

[5] More generally, for every positive integer n, one can define n-ary relation as a subset of $X^n := X \times \cdots \times X$ (n-times). We shall rarely consider n-ary relation for $n \neq 2$ and so by relation from now on we shall mean a binary relation unless otherwise specified.

(i) $[x]=[y]$. (ii) $[x]\cap[y]\neq\emptyset$. (iii) $(x,y)\in R$.

(3) (Quotient set of an equivalence relation) The set of equivalence classes in X under the relation R is denoted by X/R (read : "X modulo R") and is called the quotient set of X with respect to R. The canonical map $\pi : X \to X/R$, $x \mapsto [x]_R$ is clearly surjective and is called canonical projection of X onto X/R. The fibres of the canonical projection are precisely the equivalence classes (in X) under R. An element $x\in X$ is called a representative of the equivalence class $[x]_R$; any other element $y\in$ is a representative of $[x]_R$ if and only if $y\in[x]_R$ or equivalently $(x,y)\in R$. A (full) representative system for the quotient set X/R is a family x_i, $i\in I$ of elements in X such that the map $I\to X/R$ defined by $i\mapsto[x_i]$ is bijective, i. e., every equivalence class in X is represented by a unique element x_i, $i\in I$. In particular, a subset $X\subseteq X$ is a representative system for X/R if and only if the restriction $\pi|X' : X'\to X/R$ of the canonical projection to X' is bijective.

(b) The restriction of the map $\alpha : \mathfrak{P}(X\times X)\to\mathfrak{P}\big(\mathfrak{P}(X)\big)$, $R\mapsto\big\{\{y\in X \mid xRy\}\mid x\in X\big\}$ is injective on the subset $\mathfrak{Eq}(X)\subseteq\mathfrak{P}(X\times X)$ of all equivalence relations on X.

(c) On the set $\mathbb{N}^*$ of poistive natural numbers, let $|$ be the divisibility relation, i.e., $x \mid y$ if and only if x is a divisor of y. What is the inverse relation $|^{-1}$ on $\mathbb{N}$? Show that $|$ is an order on $\mathbb{N}^*$ and 1 is the smallest element. The minimal elements (with respect to $|$) in $\mathbb{N}^*-\{1\}$ are precisely the prime numbers.

(d) Let $f : X\to Y$ be a map. The relation $\sim$ defined by $x\sim y$ if and only if $f(x)=f(y)$ is an equivalence relation on X. The equivalence classes with respect to $\sim$ are precisely the *non-empty* fibres of f.

(e) Suppose that X is a finite set with n elements. How many reflexive (respectively, , symmetric, reflexive and symmetric) relations on X are there? (**Ans :** $2^{n(n-1)}$, $2^{\binom{n+1}{2}}$ and $2^{\binom{n}{2}}$.)

(f) (Congruence relations) Let $n\in\mathbb{N}^+$ be a positive natural number Two integers a and b are called congruent modulo n, if their difference is divisible by n. In this we write $a\equiv b \mod n$ or $a\equiv b\,(n)$. This relation on the set of integers $\mathbb{Z}$ is an equivalence relation. Two integers are congruent modulo n if and only if their remainders (between 0 and $n-1$) after the division by n are equal. Therefore the numbers $0,\ldots,n-1$ form a full representative system for the quotient set $\mathbb{Z}/\equiv$; there are exactly n equivalence classes these are called the residue classes modulo n. The set of these residue classes is usually denoted by $\mathbb{Z}/\mathbb{Z}n$. In the case $n=2$, the residue class $\bar{0}=[0]$ is the set of all even integers and the residue class $\bar{1}=[1]$ is the set of all odd integers.[6] More generally, For a real number $T\neq 0$, the relation on R defined by $a\equiv b \mod T$ or $a\equiv b\,(T)$ if the difference $b-a$ is an *integral* multiple of T, is an equivalence relation on $\mathbb{R}$. For $a\in\mathbb{R}$, the equivalence class $\bar{a}=a+\mathbb{Z}T$ of a is precisely the set of elements $a+kT$, $k\in\mathbb{Z}$. The real numbers T and $|T|$ define the same relation. The numbers in the interval $[0,|T|[\,:=\{x\in\mathbb{R}\mid 0\le x<|T|\}$ form a full representative system for the quotient set $\mathbb{R}/\mathbb{Z}T$. The unique representative of the equivalence class $\bar{a}=a+\mathbb{Z}T$ in $[0,|T|[$ is $a-[a/|T|]\cdot|T|$, where $[-]$ denote the Gauss-bracket. If $T=n\in\mathbb{N}^*$, then $\mathbb{Z}/\mathbb{Z}n\subseteq\mathbb{R}/\mathbb{Z}n$ is the set of those equivalence classes which have an integral representative.

(g) On the set $X:=\{0,1,\ldots,k-1\}^{\{1,\ldots,n\}}$ of all k-ary sequences of length n define a relation $\sim$ by : $(a_1,\ldots,a_n)\sim(b_1,\ldots,b_n)$ if $a_i=b_i$ whenever $x_i\neq 0$ or 1, $i=1,\ldots,n$. For example, if $k=4$, then $012311220330\sim 112301220331$. Show that $\sim$ is an equivalence relation on X. The equivalence class with respect to $\sim$ is called the pattern of the symbols $2,3,\ldots,k-1$. Two

[6] The congruence relations were first time systematically studied by C. F. Gauss (1777-1855) in the "*Disquisitiones arithmeticae*" (1801). Carl Friedrich Gauss worked in a wide variety of fields in both mathematics and physics including number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. His work has had an immense influence in many areas.

k-ary sequences represent the same pattern of the symbols $2,3,\ldots,n$ if and only if all the symbols $2,3,\ldots,k-1$ appear exactly at the same positions in them.

(h) Let $\preceq$ be a reflexive and transitive relation on the set A. Then the relation $\sim$ defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$, is an equivalence relation on A. On the set $\overline{A}$ of the equivalence classes of A with respect to $\sim$ the relation defined by $[a] \leq [b]$ if and only if $a \preceq b$, is a well-defined relation and is an order. (**Remark :** It is to be shown in particular that the $\leq$-relationship for two equivalence classes does not depend on the representatives used for the definition. The problem to verify such independence from the choice of the representatives is typical for computation of equivalence classes.)

T4.7 (Transitive closure of a relation) For $n \in \mathbb{N}$ we define the powers R^n of R recursively as : $R_0 := \Delta_X$ and $R^{n+1} := R \circ R^n$. Then the relation $R^+ := \cup_{n=1}^{\infty} R^n$ is called the transitive closure of R, and the relation $R^* := \cup_{n=0}^{\infty} R^n$ is called the reflexive-transitive closure of R.

(a) If $x, y \in X$ then $(x,y) \in R^*$ is either $x = y$ or there exist $x_1, x_2, \ldots, x_n \in X$ such that (x, x_1), $(x_1, x_2), \ldots, (x_{n-1}, x_n)$ are all in R. (**Hint :** By induction. In fact $n \leq 2^i - 1$)

(b) If R is symmetric then so is R^*.

(c) R^+ is the smallest transitive relation containing R.

(d) R^* is the smallest reflexive and transitive relation containing R.

(e) If R is symmetric then R^* is the smallest equivalence relation containing R.

(f) Let R_1 be the symmetric closure of the reflexive-transitive closure of R and let R_2 be the reflexive-transitive closure of the symmetric closure R. Then show that $R_1 \subseteq R_2$ and give an example showing that the reverse inclusion does not hold in general.

(g) Let M be the set of all males and let F be a relation "being a father of ..." Then F is not transitive and the transitive closure of F describes the ancestor-descendant relationship among the males.

(h) Is the transitive closure of an antisymmetric relation is always antisymmetric?

(i) On $\mathbb{Z}$ let R be the relation defined by $(x,y) \in R$ if $y = x + n$ for some fixed $n \in \mathbb{Z}$. What is the equivalence relation on $\mathbb{Z}$ generated by R?

T4.8 (Relation Matrix) Let $X := \{x_1, \ldots, x_m\}$, $Y := \{y_1, \ldots, y_m\}$ be finite sets and let R be a relation from X to Y. Then R can be specified by a matrix whose rows are labled by the elements of X and whose columns are labeled by the elements of Y. In the i-th row and j-th column we write the entry 1 if $(x_i, y_j) \in R$ and 0 if $(x_i, y_j) \notin R$. This matrix is called a relation matrix of R and is usually denoted by $\mathfrak{A}(R)$.

(a) If $X = \{a,b\}$, $Y = \{c,d,e\}$ and $R = \{(a,c),(a,d),(b,e)\}$, $R' = \{(b,c),(b,d),(a,e)\}$. Then $\mathfrak{A}(R) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathfrak{A}(R') = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(b) Conversely, each $m \times n$ matrix $\mathfrak{A} = (a_{ij})$ of 0's and 1's defines a relation R from the set X to the set Y by the rule $(x_i, y_j) \in R$ if and only if $a_{ij} = 1$. Compute the matrices of the following relations :
(i) $=$ and $\leq$ on the sets $\{-1,0,1\}, \{-2,-1,0,1,2\}$.
(ii) $=$ and "negative of" on the sets $\{-1,0,1\}, \{-2,-1,0,1,2\}$.

(c) Show that the following statements are equivalent :
(i) R is both symmetric and anti-symmetric. (ii) The matrix $\mathfrak{A}(R) = (a_{ij})$ is diagonal, that is, $a_{ij} = 0$ whenever $i \neq j$. (iii) $R \subseteq \Delta_X$.

T4.9 (Bell's numbers[7]) Let X be a finite set with n elements. The number of equivalence relations on X is called the n-Bell number β_n, i. e., $|\mathfrak{Eq}(X)| = \beta_n$.

(a) The numbers β_n satisfy the recursion relations $\beta_0 = 1$ and $\beta_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \beta_k$ for all $n \in \mathbb{N}$.

(b) Let $m, n \in \mathbb{N}$ with $m \leq n$ and let $\beta_{m,n} := \sum_{i=0}^{m} \binom{m}{i} \beta_{n-i}$. Then $\beta_{0,n} = \beta_n$, $\beta_{0,n+1} = \beta_{n,n}$ and $\beta_{m+1,n+1} = \beta_{m,n} + \beta_{m,n+1}$ for all $m, n \in \mathbb{N}$ with $m \leq n$.

(c) Using the above formulas we have the following table :

n	0	1	2	3	4	5	6	7	8	9	10
β_n	1	1	2	5	15	52	203	877	4140	21147	115975

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T4.10 (Partitions of a set) Let X be a set. A partition or decomposition $\mathfrak{P}$ of the set X is a subset $\mathfrak{P} \subseteq \mathfrak{P}(X)$ of non-empty disjoint subsets of X such that their union is $\bigcup_{Y \in \mathfrak{P}} Y = X$. In particular, a partition $\mathfrak{P}$ of X is an element of the set $\mathfrak{P}(\mathfrak{P}(X))$. More generally, an arbitrary family X_i, $i \in I$ of non-empty pairwise disjoint subsets X_i of X with $\cup_{i \in I} X_i = X$ is called a partition of X (parametrized by the index set I); in this we write $X = \uplus_{i \in I} X_i$. If $X = \cup_{i \in I} X_i$ without necessarily the condition of pairwise disjointness of X_i, $i \in I$, then the family X_i, $i \in I$, is called the covering of X.

(a) *The partition X_i, $i \in I$ of X corresponds to the surjective map $f : X \to I$.* (The partition X_i, $i \in I$, defines the map $f(x) := i$, if $x \in X_i$ and conversely the map f defines the partition $X_i := f^{-1}(i)$, $i \in I$, of X.) Therefore partitions are the fibres of the surjective maps. If X is a finite set, then clearly every partition $\mathfrak{P}$ of X is finite a finite set and $|\mathfrak{P}| \leq |X|$.

The set of all partitions of X is denoted by $\mathfrak{Par}(X)$; this is a subset of the set $\mathfrak{P}(\mathfrak{P}(X))$. As usual for $n \in \mathbb{N}$, we put $\mathfrak{Par}_n(X) = \{\mathfrak{P} \in \mathfrak{Par}(X) \mid |\mathfrak{P}| = n\}$. Clearly the family $\mathfrak{Par}_n(X)$, $n \in \mathbb{N}$ is pairwise disjoint and $\cup_{n \in \mathbb{N}} \mathfrak{Par}_n(X) = \mathfrak{Par}(X)$.

(b) The map $\alpha : \mathfrak{P}(X \times X) \to \mathfrak{P}(\mathfrak{P}(X))$, $R \mapsto \{\{y \in X \mid xRy\} \mid x \in X\}$ (see test-Exercise T4.?-(?)) maps $\mathfrak{Eq}(X)$ bijectively onto $\mathfrak{Par}(X)$, i.e. to each equivalence relation R on X, α associates a unique partition $\alpha(R)$ of X and conversely. The partition corresponding to the equivalence relation R on X is denoted by $\mathfrak{P}_R$ and the equivalence relation corresponding to the partition $\mathfrak{P}$ is denoted by $R_{\mathfrak{P}}$, i.e., the maps $\mathfrak{P}(X) \to \mathfrak{Eq}(X)$, $\mathfrak{p} \mapsto \mathfrak{p}_R$ and $\mathfrak{Eq}(X) \to \mathfrak{Par}(X)$, $R \mapsto R_{\mathfrak{p}}$ are bijective and are inverses of each other. Moreover, if $\mathfrak{Eq}_r(X)$ is the set of all equivalence relations on X with exactly r equivalence classes. Then $|\mathfrak{Eq}_r(X)| = |\mathfrak{Par}_r(X)|$ and $\mathfrak{Eq}(X) = \uplus_{r=0}^{n} \mathfrak{Eq}_r(X)$.

(c) What are the coarest and the finest partitions of a given set X? What are the corresponding equivalence relations? What are the partitions corresponding to the equivalence relations Δ_X and $X \times X$?

[7] Eric Temple Bell (1883-1960) was a Scottish mathematician and attended Bedford Modern School where excellent mathematics teaching gave him his life-long interest in the subject. In particular, his interest in number theory came from this time. Bell wrote several popular books on the history of mathematics. He also made contributions to analytic number theory, Diophantine analysis and numerical functions. The American Mathematical Society awarded him the Bôcher Prize in 1924 for his memoir, Arithmetical paraphrases which had appeared in the Transactions of the American Mathematical Society in 1921. Although he wrote 250 research papers, including the one which received the Bôcher Prize, Bell is best remembered for his books, and therefore as an historian of mathematics. Bell did not confine his writing to mathematics and he also wrote sixteen science fiction novels under the name John Taine.