E0 221 Discrete Structures / August-December 2012

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

1-st Midterm : Saturday, September 22, 2012; 14:00 -16:30 **2-nd Midterm :** Sunday, October 14, 2012; 10:00 -12:00

Final Examination : December ??, 2012, 10:00 -13:00

Evaluation Weightage : Midterms (Two) : 50% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

TEST 1

Saturday, September 22, 2012 **14:00 to 16:30** **Maximum Points :** 50 Points

- **Question T1.6 is C O M P U L S O R Y.**
- **Attempt O N L Y F I V E Questions.**

T1.1 **(a)** Let A, B, C and D be sets. If $A \subseteq C$ and $B \subseteq D$, then show that $A \times B \subseteq C \times D$. Moreover, if $A \neq \emptyset$ and $B \neq \emptyset$, then the converse also holds. [3 points]

(b) Investigate whether the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $(x,y) \mapsto (xy, x+y)$, is injective, surjective respectively, bijective. [4 points]

(c) Draw the pictures of the fibres $f^{-1}(1)$ and $g^{-1}(-1)$ of the maps $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto xy$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto |y-x|$, at the points 1 and -1, respectively. [4 points]

T1.2 Let $f : X \to Y$, $g : Y \to X$ and $h : X \to Y$ be maps. Show that:

(a) If $g \circ f$ is bijective, then f injective and g is surjective. Give an example to show that neither f nor g is bijective even if $g \circ f$ is bijective. [4 points]

(b) If $g \circ f$ is bijective and if f is bijective, then g is also bijective. [3 points]

(c) From the equalities $g \circ f = \mathrm{id}_X$ and $h \circ g = \mathrm{id}_Y$, show that equality $f = h$. Further, show that g is bijective and $g^{-1} = f = h$. [4 points]

T1.3 Let X and Y be sets.

(a) Show that the map $\Gamma : \mathrm{Maps}(X,Y) \to \mathfrak{P}(X \times Y)$ defined by $f \mapsto \Gamma_f := \{(x, f(x)) \mid x \in X\}$ the graph of f is injective. [5 points]

(b) For $X := [0,1] := \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$ and $Y := \mathbb{R}$, find the fibres $\Gamma^{-1}(R)$ and $\Gamma^{-1}(S)$ over the relations $R := \{(x,y) \in X \times Y \mid x < y\}$ and $S := \{(x,1) \in X \times Y \mid x \in X\}$ under the map Γ given in the part (a) above. [5 points]

T1.4 **(a)** On the set $\mathbb{N}^+$ of positive natural numbers, let $\mid$ denote the relation "divides", i. e. for $m, n \in \mathbb{N}^+$, $m \mid n$ if and only if $n = am$ for some $a \in \mathbb{N}^+$. Show that:

(i) $\mid$ is an order on $\mathbb{N}^+$ and that the element 1 is the least element. [2 points]

(ii) The prime numbers are precisely the minimal elements in $(\mathbb{N}^+ \setminus \{1\}, \mid)$. [2 points]

(iii) Draw the Hasse-Diagrams for the set of divisors of 12 and 30. [2 points]

(iv) The subset $\mathrm{C} := \{2^n \mid n \in \mathbb{N}\} \subseteq \mathbb{N}^+$ is a maximal chain in the ordered set $(\mathbb{N}^+, \mid)$. [3 points]

(b) Give an example of an ordered set $(X, \leq)$ such that there are exactly three minimal elements and two maximal elements and neither a minimum nor a maximum. [2 points]

T1.5 (a) Let $(X, \leq)$ be a conditionally complete ordered set and $f : X \to X$ be a non-decreasing map from X to X. If there are $a, b \in X$ such that $a \leq f(a) \leq f(b) \leq b$, then there exists an element $c \in X$ such that $a \leq c \leq b$ and $f(c) = c$. [5 points]

(b) Explain why one cannot apply the result in part (a) above to the map $f : (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$, $n \mapsto n^2 + 1$, even though the map f is non-decreasing? [2 points]

(c) Give an example of an order $\preceq$ on the set $\mathbb{N}$ of natural numbers such that the ordered set $(\mathbb{N}, \preceq)$ is a complete ordered set. [3 points]

***T1.6 (a)** Let $(M, \cdot)$ be a monoid with neutral element e. For an element a in a monoid M, show that the following statements are equivalent :

(i) a is invertible in M.

(ii) The left translation $\lambda_a : M \to M$, $x \mapsto a \cdot x$ is bijective.

(iii) The right translation $\rho_a : M \to M$, $x \mapsto x \cdot a$ is bijective. [4 points]

(b) Is the successor map $\sigma : \mathbb{N} \to \mathbb{N}$, $n \mapsto n+1$, invertible in the monoid $(\mathbb{N}^{\mathbb{N}}, \circ)$? [2 points]

(c) Give at least two explicit examples of invertible elements in the monoid $(\mathbb{N}^{\mathbb{N}}, \circ)$. Is the set $(\mathbb{N}^{\mathbb{N}}, \circ)^{\times}$ of invertible elements in the monoid $(\mathbb{N}^{\mathbb{N}}, \circ)$ finite? [4 points]

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