E0 221 Discrete Structures / August-December 2013

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 10:00–11:30   **Venue:** CSA, Lecture Hall (Room No. 117)

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Quizzes : During Wednesday, Lectures on **Aug 28; Sept 18; Oct 09; Oct 30;**   **Time :** 10:00 -10:15
1-st Midterm : Saturday, September 14, 2013; 14:00 -16:30   **2-nd Midterm :** Saturday, October 12, 2013; 10:00 -12:00
Final Examination : ????????, December ??, 2013, 14:00 -17:00

Evaluation Weightage : Quizzes (Four) + Midterms (Two) : 50%   **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

1. Sets – Operations on Sets

1.1 For a set A, show that the following statements are equivalent:

(i) $A=\emptyset$.
(ii) $A\setminus B = A\cap B$ for every set B.
(iii) There exists a set B with $A\setminus B = A\cap B$.
(iv) $B\setminus A = B\cup A$ for every set B.
(v) There exists a set B with $B\setminus A = B\cup A$.

1.2 Show that the operation $(\cdot\setminus\cdot)$ is distributive over the operations $\cup$ and $\cap$, i. e. for arbitrary three sets A,B,C :
(a) $(A\cup B)\setminus C = (A\setminus C)\cup(B\setminus C)$.
(b) $(A\cap B)\setminus C = (A\setminus C)\cap(B\setminus C)$.

1.3 Show that $(A\cap B)\cap(A\setminus B)=\emptyset$ and $(A\cap B)\cup(A\setminus B)=A$.

1.4 Prove that the statements $A\subseteq B \iff (C\setminus B)\subseteq(C\setminus A)$ and $(C\setminus(C\setminus A))=A$ are not true for some sets A, B and C.

1.5 For non-empty sets A and B, show that the following statements are equivalent:
(i) $A\times B\subseteq B\times A$. (ii) $B\times A\subseteq A\times B$. (iii) $A\times B = B\times A$. (iv) $A=B$.
(**Remark :** This shows that $\times$ is not a commutative operation.)

1.6 Let A, B, C and D be sets.
(a) If $A\subseteq C$ and $B\subseteq D$, then show that $A\times B\subseteq C\times D$. Moreover, if $A\neq\emptyset$ and $B\neq\emptyset$, then the converse also holds.
(b) $(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup D)$. Moreover, the equality holds if and only if the conditions "$A\subseteq C$ or $D\subseteq B$" and "$C\subseteq A$ or $B\subseteq D$" are satisfied.

1.7 Let A be a set and let $\mathfrak{P}(A)$ be the power set of A. Show that
(a) If $B\in\mathfrak{P}(A)$, then $\mathfrak{P}(B)\subseteq\mathfrak{P}(A)$.   **(b)** $\bigcap_{B\in\mathfrak{P}(A)} B=\emptyset$.

(c) Let I be a set whose members are also sets. Then

$$\bigcap_{A\in I}\mathfrak{P}(A)=\mathfrak{P}(\cap_{A\in I}A) \text{ and } \bigcup_{A\in I}\mathfrak{P}(A)\subseteq\mathfrak{P}(\cup_{A\in I}A)\,.$$

Moreover, the last inclusion is, in general, not an equality.

Below one can see Lecture Notes.

Lecture Notes

Though the naive or intuitive approach to sets will suffice for most purpose, an exposition of general set theory requires more precision, for without explicit axioms, it is well known that various contradictions arise. It was the famous English philosopher Bertrand Russell[1] (1872-1970) who shook the Mathematics community in 1901 by declaring the admission of a set of all sets would lead to a contradiction. This is the famous Russell's Paradox[2]:

T1.1 Russell's Paradox : Serious difficulties occur if we allow the notion of set to be too general. For example, it is undesirable to talk of "the set U of all sets." If such a set exists, it must, being a set, be a member of itself, that is, $U \in U$. This is not in itself disastrous, but now consider Russell's famous set V of *all sets which are not members of themselves.* (Russell's argument is to be compared with the ancient paradoxes of the "liar" type, which were the subject of innumerable commentaries in classical formal logic; the question whether a man who says "I am lying" is telling the truth or not when he speaks these words.) If V is not a member of itself then by that fact it qualifies as a member of V, that is, it is then a member of itself and the situation is no better if we yield to this reasoning and allow that V is a member of itself. For then V does not qualify as a member of V, which only admits as members those sets which are not members of themselves, so we find that we have again proved the opposite of what we have assumed.

Therefore, we must accept the terms "set" and "element" as undefined terms or primitives and guide these primitives by a number of axioms. It is desired to indicate only a framework within which we will work, which avoids the known antinomies and which, at least until now, has not led to any contradiction.

T1.2 Classical Logic : In what follows, we are assuming that we are working with *classical logic*. To prevent any misunderstanding, ambiguity or arbitrary interpretation, the essential definitions as well as the axioms of the theory of sets are introduced below using logical connectives:

(1) $\vee$ ("or") (in the sense "one or the other *or both*"),

(2) $\neg$ ("not"),

(3) $\exists$ ("there exists" or "for some"),

(4) ($\wedge$ ("and"),

[1]Bertrand Russell was born on May 18, 1872, at Trelleck, Wales. before he was four both of his parents died. He had been a shy, silent boy until he entered Trinity College, Cambridge University in 1880. After three years of Mathematics he concluded that what he was being taught was full of errors. He sold of his Mathematics books and changed to philosophy. In his *Pricipia Mathematica* (1910-1913), a three volume monumental work co-authored with Alfred North Whitehead (1861-1947), he attempted to recast set theory so as to avoid paradoxes. In 1918 he wrote, "I want to stand at time rim of the world and peer into the darkness beyond and see a little more than others have seen .... I want to bring back into world of men some little bit of wisdom." He certainly did, more than just "some little bit." In the same year he was put into prison for an unfavorable comment about the American Army. In 1950 he received the Order of Merit from King of England and the Nobel prize for literature. In his later years he led a number of demonstrations against nuclear warfare.

[2]Russell's Paradox was not the only one to arise in set theory. Shortly after the Russell's Paradox appeared many paradoxes were constructed by several mathematicians and logicians. This precipitated a search for a rigorous foundation of set theory which would avoid contradictions. As a consequence of all these paradoxes, many mathematicians and logicians (with David Hilbert (1862-1943) among its leaders) have contributed to several brands such as "*Zermelo-Fraenkel-Skolem axiomatic set theory*" (proposed by Ernst Friedrich Ferdinand Zermelo (1871-1953), Adolf Abraham Halevi Fraenkel (1891-1965) and Thoralf Albert Skolem (1887-1963)) and "*von Neumann-Bernays-Gödel axiomatic set theory*." (proposed by John von Neumann (1903-1957), Paul Isaac Bernays (1888-1977) and Kurt Gödel (1906-1978)) of "axiomatic set theory," each designed to avoid these paradoxes and at the same time to preserve the main body of Cantor's set theory.

(5) $\forall$ ("for every"),

(6) $\Rightarrow$ ("implies"),

(7) $\iff$ ("if and only if"), but we are rather casual about this.

T1.3 The Role of Definitions : A very important ingredient of mathematical creativity is the ability to formulate useful definitions — the ones that will lead to interesting results. Every definition is understood to be an *if and only if* type of statement — even though it is customary to suppress *only if*. For example, one may define : "A triangle is isosceles if it has two sides of equal length" The important thing to *understand* the concept so that you can define precisely the same concept in your words. The above definition may also be reformulated as: "An isosceles triangle is one having two equal angles"

It is very important to note that proofs concerning the concepts just been defined *must* use the definition as an integral part of the proof because *the definition is the only information available regarding the concept immediately after it is defined*.

Throughout these lecture notes a term that appears in "gesperrt" type is being *defined* defined at that point.

T1.4 Sets : A set is a collection of well-defined objects[3] – meaning that if A is a set and a is some object then either a is definitely in A or a is definitely not in A. The objects that form the set are called elements or members of the set. We denote sets by capital letters A, B, C, etc. and elements of sets are denoted by lower case letters a, b, c etc. If a is an element of a set A then we write $a \in A$ and read it as "a belongs to A." If a is not an element of a set A then we write $a \notin A$ and read it as "a does not belong to A."

The logical impasse can be avoided by restricting the notion of set, so that "very large" collections or the "collection of all things" are not counted as sets. We shall never need to deal with any sets large enough to cause trouble in this way and consequently we may put aside all such worries and hope that paradoxes will not appear. However, our main interest in this course is the application of the theory of sets to the basic notions of mathematics. For example, in formulating fundamental notions such as relations, functions, natural numbers, integers, rational numbers, real numbers, ordinal numbers and cardinal numbers as well as their arithmetic [4]

Therefore we shall take a naive, non-axiomatic approach of set theory. In fact the entire discussion may be made rigorously precise.

T1.5 Some Examples : Sets will be described by explicitly declaring its elements — one calls this as enumerative notation, — or by giving characterization of its elements by means of a property $P(x)$ and is written in the brace notation $\{x \in P(x)\}$ and is read "the set of all x such that the statement $P(x)$ about x is true". For example,

$$\{1, -1\} = \{1, -1, 1\} = \{x \mid x \text{ is a real number and if } x^2 = 1\} = \{x \in \mathbb{R} \mid x^2 = 1\}$$

One and the same set can be described in many ways. For the equality of sets it is only important that they have the same elements.

We shall use the following standard notation:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ The set of natural numbers.

[3]For example, one should never say "consider the set A of some students from IISc who are registered for the Discrete Structures course". For it is not definite whether "John$\in A$" or John$\notin A$. However, since every positive integer is definitely either a prime number or not a prime number, one can consider the set $\mathbb{P}$ of all positive prime numbers. It may be hard to determine whether a given object is in a set. For example, it is unknown whether $2^{2^{17}} + 1$ is in the set $\mathbb{P}$. However, it is certainly either prime or not prime.

[4]The arithmetic of ordinal and cardinal numbers is also called transfinite arithmetic.

- $\mathbb{N}^* = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$ The set of positive natural numbers.
- $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ The set of integers.
- $\mathbb{Q} = \{\frac{a}{b} = a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$ The set of rational numbers.
- $\mathbb{R}$ The set of real numbers.
- $\mathbb{R}^\times = \{x \in \mathbb{R} \mid x \neq 0\}$ The set of non-zero real numbers.
- $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ The set of non-negative real numbers.
- $\mathbb{R}_- = \{x \in \mathbb{R} \mid x \leq 0\}$ The set of non-positive real numbers.
- $\mathbb{R}_+^\times = \{x \in \mathbb{R} \mid x > 0\}$ The set of positive real numbers.
- $\mathbb{C} = \{x + \mathrm{i}y \mid x, y \in \mathbb{R}\}$ The set of complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} \mid z \neq 0\}$ The set of non-zero complex numbers.

T1.6 (Operations on Sets) We collect some important operations os sets. Let A, B and C be arbitrary sets.

(1) The set B is called a subset of A (or A is called a supset of B) if every element of B is an element of A. The notations $B \subseteq A$ or $A \supseteq B$ are both used to mean "B is a subset of A." or "A is a supset of B". The symbol $\subseteq$ is called the inclusion of the set B in in the set A. Moreover, if $B \neq A$, then we also write $B \subset A$ or $B \subsetneq A$ and say that B is a proper subset of A (or A is a proper supset of B). (**Remarks :** The inclusion relation $\subseteq$ is not to be confused with the membership relation $\in$. For example, $\emptyset \subseteq \emptyset$, but not $\emptyset \in \emptyset$; $\{\emptyset\} \in \{\{\emptyset\}\}$ but $\{\emptyset\} \nsubseteq \{\{\emptyset\}\}$ because there is a member of $\{\emptyset\}$, namely, $\emptyset$, that is not a member of $\{\{\emptyset\}\}$; Let US be the set of all people in the United States and let UN be the set of all countries belonging to the United Nations. Then John Jones $\in$ US $\in$ UN, but John Jones $\in$ UN (since he is not even a country), and hence US $\nsubseteq$ UN.)

(2) There is exactly one set with no elements and is called the empty (or null or vacuous)-set which is usually denoted[5] by $\emptyset$ and is a subset of every set.

(3) (Union) There is a unique set $A \cup B$ such that $x \in A \cup B$ if and only if either $x \in A$ or $x \notin B$. In symbols: $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$ and is called the union of the sets A and B. (**Remark :** By repeating this operation we can form the union of three sets, four sets etc. moreover, form the union of finitely many sets. But suppose we want to form the union of *infinitely many* sets, then we need a more general union operation. This leads us to the following definition: For any set I (whose members are sets), the set of all the elements of all the members of I is called the union-set or the sum-set of I. In symbols: in the case that $I = \{A \mid A \in I\}$, the union-set of I $\{x \mid x \in A \text{ for some } A \in I\}$ is usually denoted by $\bigcup_{A \in I} A$. For example, if $I = \{\emptyset, \{\emptyset\}\}$, then $\bigcup_{A \in I} A = \{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$. However, $\bigcup_{A \in \emptyset} A = \bigcup_{A \in \{\emptyset\}} A = \emptyset$. The operation $\cup$ is idempotent, commutative and associative, see Test-Exercise T1.7.)

(4) (Intersection) There is a unique set $A \cap B$ such that $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. In symbols: $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$ and is called the intersection of the sets A and B. (**Remark :** As in the case of union, we also need the corresponding generalization of the intersection operation. In general, we define for every non-empty set I, the subset of $\bigcup_{A \in I} A$ consisting of all common elements of all the members of I is called the intersection-set of I and is denoted by $\bigcap_{A \in I} A$. In symbols: $x \in \bigcap_{A \in I} A$ if and only if $x \in A$ for every $A \in I$. In contrast to the union operation, there is no special axiom is needed to justify the intersection operation. However, there is one trouble extreme case, namely, what happens if $I = \emptyset$?. There is no set C such that for every x, $x \in C$ if and only if x belongs to every member of $\emptyset$, since the right hand side is true for every x. This presents a mild notational problem: How to

[5]The symbol $\emptyset$ is not the Greek letter phi ϕ, but rather a letter of Danish and Norwegian alphabets. The symbols $\Box$ and $\wedge$ also appear in literature for $\emptyset$.

define $\bigcap\emptyset$? One option[6] is to leave $\bigcap\emptyset$ undefined, since there is no very satisfactory way of defining it! For example, if $I = \{\emptyset, \{\emptyset\}\}$, then $\bigcap_{A\in I} A = \{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$. The operation $\cap$ is idempotent, commutative and associative, see Test-Exercise T1.7. For non-emptyset I, clearly $\bigcap_{A\in I} A \subseteq \bigcup_{A\in I} A$. Further, note that the union-set $\bigcup_{A\in I} A$ of the set $I = \{A \mid A \in I\}$ is the "smallest" set which includes all the sets $A \in I$ and the intersection-set $\bigcap_{A\in I} A$ of the set I is the "largest" set which is a subset of every set $A \in I$.)

(5) (Difference) There is a unique set $A \setminus B$ such that $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$. In symbols: $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$ and is called the *difference* of the sets A and B. If B is a subset of A, then the difference set $A \setminus B$ is also called the *complement of B in A* and is usually denoted by $\complement_A B$. (**Remark :** Note that for every set A, $A \setminus \emptyset = A$, $\emptyset \setminus A = \emptyset$ and $A \setminus A = \emptyset$. Therefore the difference operation $(\cdot \setminus \cdot)$ is not commutative. Further, since $(A \setminus \emptyset) \setminus A = \emptyset$ and $A \setminus (\emptyset \setminus A) = A$, it is also not associative. It is interesting to note that the inclusion and intersection can be expressed in terms of the difference: $A \subseteq B$ if and only if $A \setminus B = \emptyset$; and $A \cap B = A \setminus (A \setminus B)$.)

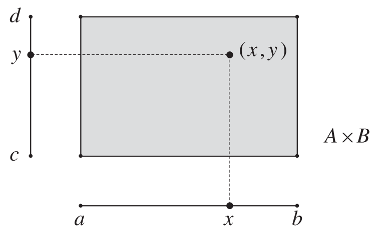
(6) (Symmetric Difference) The *symmetric difference* $A \triangle B$ of the sets A and B is the set of all those elements that are elements of A or B but not of both. In symbols: $A \triangle B = (A \cup B) \setminus (A \cap B)$. Clearly, $A \triangle B = \{x \mid \text{either } x \in A \text{ or } x \in B\} = (A \setminus B) \cup (B \setminus A)$ which justifies the term the symmetric difference. (**Remark :** The symmetric difference operation is commutative and nilpotent, i. e. $A \triangle B = B \triangle A$ and $A \triangle A = \emptyset$. Moreover, it is associative, see also Exercise 1.??.)

(7) (Power-Set) There is a unique set $\mathfrak{P}(A)$ whose elements are precisely all subsets of A. This set is called the *power-set* of A. Note that[7] unions, intersections, differences and symmetric differences of the sets from $\mathfrak{P}(A)$ are again members of $\mathfrak{P}(A)$. For example, the power-set $\mathfrak{P}(\{1,2,3\})$ of the set $\{1,2,3\}$ is the set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Note that the power-set $\mathfrak{P}(\emptyset) = \{\emptyset\}$ of the empty-set $\emptyset$ is a non-empty set.

(8) (Cartesian-product) For sets A and B, the set of (ordered) pairs (a,b), $a \in A$ and $b \in B$, is called the *Cartesian-product* or the *cross-product* of the sets A and B and is usually denoted by $A \times B$. (**Remark :** The set $\{\{a\}, \{a,b\}\}$ is called the *ordered pair* and is denoted by (a,b). The two ordered pairs (a,b) and (a',b') are equal if and only if the corresponding *components* are equal, i. e. $a = a'$ and $b = b'$. Therefore the pair (a,b) and the set $\{a,b\}$ needs to be distinguished! For $a \neq b$, naturally $(a,b) \neq (b,a)$ but $\{a,b\} = \{b,a\}$.)

For example, let $A := [a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ and $B := [c,d] = \{y \in \mathbb{R} \mid c \leq y \leq d\}$ be two closed intervals with $a < b$ and $c < d$. Then $A \times B$ is the "rectangle":

$$[a,b] \times [c,d] = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid a \leq x \leq b\,,\, c \leq y \leq d\}\,.$$

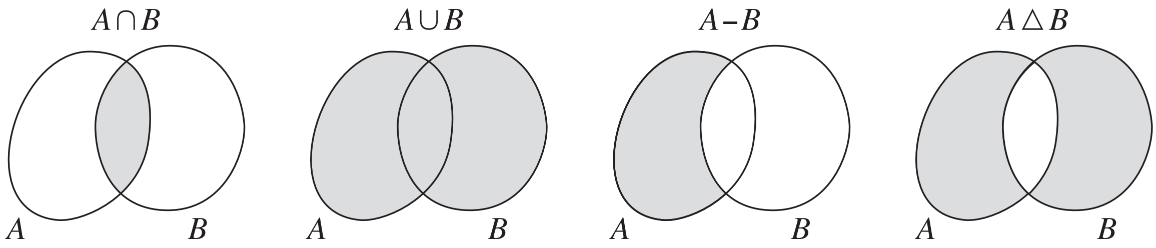


T1.7 (Euler-Venn Diagrams[8]) Euler-Venn diagrams are drawings that illustrate abstract ideas. Generally circles are drawn so that they overlap to illustrate set theory concepts.

[6] This option works perfectly well, but some logicians dislike it. It leaves the intersection-set $\bigcap_{A\in\emptyset} A$ of the empty-set $\emptyset$ as an untidy loose end, which they may later trip over!

[7] Therefore $\cup$, $\cap$, $(\cdot \setminus \cdot)$ and $\triangle$ are binary operations on the power-set $\mathfrak{P}(A)$.

[8] *John Venn (1834-1923)* was an English logician who made contributions to logic and probability. He was an ordained minister but resigned his ministry in 1883 to concentrate on logic, which he taught at Cambridge. The diagrams for which he remembered were actually used earlier by a Swiss mathematician *Leonhard Euler* (1707-1783), but were perfected by Venn.



T1.8 (Algebra of Sets[9] and Computational-Rules) The following identities which hold for arbitrary sets, are some computational rules in the algebra of sets:

Let A,B and C be arbitrary sets. Then:

(1) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(2) $A \cup A = A$ and $A \cap A = A$. (Idenpotency)

(3) $A \cup B = B \cup A$ and $A \cap B = B \cap A$. (Commutativity)

(4) $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. (Associativity)

(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (Distributivity)

(6) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. (De Morgan's laws[10])

T1.9 For arbitrary three sets A, B, and C, show that:
(a) $(A \setminus B) = A$ if and only if $(B \setminus A) = B$.
(b) $B = \emptyset$ if and only if $A \cup B = A \setminus B$.
(c) $A = B$ if and only if $(A \setminus B) = (B \setminus A)$.
(d) $A \subseteq B \cup C$ if and only if $(B \setminus C) \subseteq A$.
(e) $(B \setminus A) \subseteq C$ if and only if $(B \setminus A) = B$.

T1.10 For sets A and B, show that the following statements are equivalent:

(i)	$A \subseteq B$.	(ii)	$A \cap B = A$.
(iii)	$A \cup B = B$.	(iv)	$A \setminus B = \emptyset$.
(v)	$B \setminus (B \setminus A) = A$.	(vi)	$A \cup (B \cap C) = (A \cup C) \cap B$ for every set C.

(vii) There exists a set C with $A \cup (B \cap C) = (A \cup C) \cap B$.

T1.11 For sets A, B, C, show that:
(a) $A \setminus (B \cup C) = (A \setminus B) \setminus C$.
(b) $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C) = (A \cap C) \setminus B$.
(c) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.
(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
(e) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
(f) $A \cup (B \setminus C) \supseteq (A \cap B) \setminus (A \cap C)$. Further, show that the equality holds if and only if $A \cap C = \emptyset$ and give an example to show that, in general, the equality does not hold.

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[9] The study of operations of union $\cup$, intersections $\cap$, differences $(\cdot \setminus \cdot)$ together with the inclusion $\subseteq$ goes by the name algebra of sets. In some ways algebra of sets obeys laws reminiscent of algebra of real numbers (with $+$, $\cdot$, $-$ and $\leq$), but there are significant differences!

[10] Augustus De Morgan (1806-1871) was an English logician who made major contributions to logic and probability. De Morgan was a brilliant mathematician who introduced the slash notation for representing fractions, such as $1/2$ and $3/4$. Once asked when he was born, De Morgan replied, "I was x years old in the year x^2." Can you determine the year he was born ?

T1.12 For sets A,B,C, show that:

(a) $A \triangle A = \emptyset$ and $A \triangle \emptyset = A$.
(b) $A = B$ if and only if $A \triangle B = \emptyset$.
(c) $A \cap B = \emptyset$ if and only if $A \triangle B = A \cup B$.
(d) $(A \triangle B) \cap (A \cap B) = \emptyset$.
(e) $(A \triangle B) \cup (A \cap B) = A \cup B$.
(f) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.
(g) If $A \triangle B = A \triangle C$, then $B = C$.
(h) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(**Remark :** For neater proof use the Exercise T2.27.)

T1.13 For arbitrary four sets A, B, C and D, show that:
(a) $(A \setminus B) \cup C = ((A \setminus (B \setminus C)) \cup (C \setminus A)$.
(b) $A \cup (B \setminus C) = ((A \cup B) \setminus C) \cup (A \cap C)$.
(c) $(A \setminus B) \cup C = ((A \cup C) \setminus B) \cup (B \cap C)$.
(d) $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$.
(e) $(A \setminus B) \setminus (C \setminus D) = (A \setminus (B \cup C)) \cup ((A \cap D) \setminus B)$.
(f) $A \setminus (B \setminus (C \setminus D)) = (A \setminus B) \cup ((A \cap C) \setminus D)$.
(g) $A \setminus (A \setminus (B \setminus (B \setminus C))) = A \cap B \cap C$.
(h) $(A \setminus D) \subseteq (A \setminus B) \cup (B \setminus C) \cup (C \setminus D)$.

T1.14 Show that the operation $\times$ is distributive over the operations $\cup$, $\cap$ and $(\cdot \setminus \cdot)$, i. e. for arbitrary three sets A,B,C :
(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(B \cup C) \times A = (B \times A) \cup (C \times A)$.
(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ and $(B \cap C) \times A = (B \times A) \cap (C \times A)$.
(c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ and $(B \setminus C) \times A = (B \times A) \setminus (C \times A)$.

T1.15 For sets A,B,C,D, show that:
(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(b) If $A \subseteq C$ and $D \subseteq B$, then show that $(C \times D) \setminus (A \times B) = ((C \setminus A) \times D) \cup (C \times (D \setminus B))$.

T1.16 Let A_i, $i \in I$, and let B_j, $j \in J$, be families of sets with $I \neq \emptyset \neq J$. Then:
(a) $\big(\bigcap_{i\in I} A_i\big) \cup \big(\bigcap_{j\in J} B_j\big) = \bigcap_{(i,j)\in I\times J} (A_i \cup B_j)$, $\big(\bigcap_{i\in I} A_i\big) \cup \big(\bigcup_{j\in J} B_j\big) = \bigcap_{i\in I} \big(\bigcup_{j\in J} A_i \cup B_j\big)$.
(b) $\big(\bigcup_{i\in I} A_i\big) \cap \big(\bigcup_{j\in J} B_j\big) = \bigcup_{(i,j)\in I\times J} (A_i \cap B_j)$, $\big(\bigcup_{i\in I} A_i\big) \cap \big(\bigcap_{j\in J} B_j\big) = \bigcup_{i\in I} \big(\bigcap_{j\in J} A_i \cap B_j\big)$.
(c) $\big(\bigcup_{i\in I} A_i\big) \setminus \big(\bigcup_{j\in J} B_j\big) = \bigcup_{i\in I} \big(\bigcap_{j\in J} (A_i \setminus B_j)\big)$, $\big(\bigcup_{i\in I} A_i\big) \setminus \big(\bigcap_{j\in J} B_j\big) = \bigcup_{(i,j)\in I\times J} (A_i \setminus B_j)$.
(d) Further, if A is another set, then $A \setminus \big(\bigcup_{j\in J} B_j\big) = \bigcap_{j\in J} (A \setminus B_j)$ und $A \setminus \big(\bigcap_{j\in J} B_j\big) = \bigcup_{j\in J} (A \setminus B_j)$.
In particular, the complement of a union is equal to the intersection of the complements and the complement of an intersection is equal to the union of the complements.