E0 221 Discrete Structures / August-December 2013

(ME, MSc. Ph. D. Programmes)

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Tel : +91-(0)80-2293 2239/(Maths Dept. 3212)   **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 10:00–11:30   **Venue:** CSA, Lecture Hall (Room No. 117)

TAs/Corrections by : Akanksha Agrawal (akanksha.agrawal@csa.iisc.ernet.in)
Palash Dey (palash@csa.iisc.ernet.in) / **Govind Sharma** (govindjsk@csa.iisc.ernet.in)
Sayantan Mukheriee (sayantan.mukherjee@csa.iisc.ernet.in)

Quizzes : During Wednesday, Lectures on **Aug 28; Sept 18; Oct 09; Oct 30;**   **Time :** 10:00 -10:15
1-st Midterm : Saturday, September 14, 2013; 14:00 -16:30   **2-nd Midterm :** Saturday, October 12, 2013; 10:00 -12:00
Final Examination : ????????, December ??, 2013, 14:00 -17:00

Evaluation Weightage : Quizzes (Four) + Midterms (Two) : 50%   **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	>90	76–90	61–75	46–60	35–45	<35

3. Relations — Equivalence relations

3.1 Let $f:X\to Y$ be a map. The relation $\sim$ defined by $x\sim y$ if and only if $f(x)=f(y)$ is an equivalence relation on X. The equivalence classes with respect to $\sim$ are precisely the *non-empty* fibres of f.

3.2 Give examples of relations which satisfy the two of the three properties of the equivalence relations, but not the third one. How many relations are there on the set with n elements?

3.3 Let $\preceq$ be a reflexive and transitive relation on the set X. Then the relation $\sim$ defined by $a\sim b$ if and only if $a\preceq b$ and $b\preceq a$, is an equivalence relation on X. On the set $\overline{X}$ of the equivalence classes of X with respect to $\sim$ the relation defined by $[a]\leq[b]$ if and only if $a\preceq b$, is a well-defined relation and is an order. (**Remark :** It is to be shown in particular that the $\leq$-relationship for two equivalence classes does not depend on the representatives used for the definition. The problem to verify such independence from the choice of the representatives is typical for computation of equivalence classes.)

3.4 For $k\in\mathbb{N}^+$, a k-ary sequence is a sequence with values in a finite set with k elements (generally in the set $\{0,\ldots,k-1\}$), i.e. a k-ary sequence is an element in the set $\{0,\ldots,k-1\}^{\mathbb{N}}$. For $k=2,3,4,5$ these sequences are also called binary, ternary, quaternary, quintnary sequences. On the set $X:=\{0,1,\ldots,k-1\}^{\{1,\ldots,n\}}$ of all k-ary sequences of length n define a relation $\sim$ by : $(a_1,\ldots,a_n)\sim(b_1,\ldots,b_n)$ if $a_i=b_i$ whenever $a_i\neq 0$ or 1, $i=1,\ldots,n$. For example, if $k=4$, then $012311220330\sim112301220331$. Show that $\sim$ is an equivalence relation on A. The equivalence class with respect to $\sim$ is called the pattern of the symbols $2,3,\ldots,k-1$. Two k-ary sequences represent the same pattern of the symbols $2,3,\ldots,k-1$ if and only if all the symbols $2,3,\ldots,k-1$ appear exactly at the same positions in them.

3.5 What are the coarest and the finest partitions and the corresponding equivalence relations of a given set X? What are the partitions corresponding to the equivalence relations Δ_X and $X\times X$?

3.6 Let R denote the relation on the set $\mathbb{N}\times\mathbb{N}$ defined by

$$((a,b),(m,n))\in R \quad \text{if and only if} \quad a+n=b+m$$

Show that R is an equivalence relation on $\mathbb{N}\times\mathbb{N}$. For $(m,n)\in\mathbb{N}\times\mathbb{N}$, let $[m,n]:=[(m,n)]_R$ be the equivalence class of $(m,n)\in\mathbb{N}\times\mathbb{N}$ under R. What are the equivalence classes $[0,0],[0,2],[3,0]$? Further, show that:

(a) The set $\{(0,n),(0,0),(m,0)\mid m,n\in\mathbb{N}^+\}$ form a complete system of representative for the quotient set $\mathbb{N}\times\mathbb{N}/R$.

(b) The map $\mathbb{N}\times\mathbb{N}/R\to\mathbb{Z}$ defined by $[0,n]\mapsto -n$, $[0,0]\mapsto 0$ and $[m,0]\to m$, is well-defined. Moreover, show that it is bijective.

Below one can see Lecture Notes.

Lecture Notes

As a preparation to pursue more modern mathematics, this Lecture begins with a discussion on *relations* and its intimate connection between equivalence relations and partitions. An abundance of examples is provided.

Who married with whom is simply expressed by the set of married couples. More generally, we define :

T3.1 Definitions : Let A and B be sets. A (binary) relation [1] R from A and B is a subset $R \subseteq A \times B$, i.e. an element $R \in \mathfrak{P}(A \times B)$. For the expression "$(x,y) \in R$" we shall write "$xRy$" and say that *"$x$ is related to y with respect to R", $x \in A$, $y \in B$.*

For the relations defined by special properties we use special symbols. For example, the equality defined the equality-relation $=$ on the set A and is therefore the subset $\Delta_A := \{(a,a) \mid a \in A\}$, called the diagonal of A. For $x, y \in A$, $x = y$ is equivalent to $(x,y) \in \Delta_A$. The diagonal Δ_A can also be interpreted as the graph of the identity map $\mathrm{id}_A : A \to A$ of A. More generally, every map $f : A \to B$ defines a relation from A to B, namely, its graph $\Gamma_f \subseteq A \times B$.

The set of relations $\mathfrak{P}(A \times B)$ from A to B is also denoted by $\mathrm{Rel}(A,B)$ and its elements are also denoted by the symbols $\sim$, $\cong$ $\equiv$, $\leq$, $\preceq$ $\cdots$. In the case $B = A$, we put $\mathrm{Rel}(A) = \mathrm{Rel}(A,A) = \mathfrak{P}(A,A)$ and its elements are called relation on A. The relation $R = \emptyset$ and $R = A \times B$ are called the empty-relation and the all-relation from A to B, respectively. Furthermore, we can also define intersection and union of arbitrary family of relations.

Let R be a relation form a set A to a set B. The subset $\{x \in A \mid \text{there exists}\, y \in B \text{ such that } (x,y) \in R\}$ of A of all first coordinates of R is called the domain of R and is usually denoted by $\mathrm{Dom}(R)$. The subset $\{y \in B \mid \text{there exists } x \in A \text{ such that } (x,y) \in R\}$ of B of all second coordinates of R is called the range or image of R and is usually denoted by $\mathrm{Rng}(R)$ or $\mathrm{Img}(R)$. In particular, $R \subseteq \mathrm{Dom}(R) \times \mathrm{Rng}(R)$, but this inclusion may be strict (see Examples (c) and (d) below in T3.2)

T3.2 Examples : Let A and B be sets.

(a) The empty set (called the empty relation) $\emptyset$ is a relation from A to Y and $\mathrm{Dom}(\emptyset) = \emptyset = \mathrm{Rng}(\emptyset)$. The product set $A \times B$ (called the all-relation) is also a relation from A to B and if $A \times B \neq \emptyset$, then $\mathrm{Dom}(A \times B) = A$ and $\mathrm{Rng}(A \times B) = B$.

(b) The diagonal subset $\Delta_A := \{(x,x) \mid x \in A\}$ is a relation on A and is called the diagonal or identity relation on A.

(c) The subset $\{(x, A') \in A \times \mathfrak{P}(A) \mid x \in A'\}$ is a relation from A to $\mathfrak{P}(A)$, called the element-hood relation, its domain is A and range is $\mathfrak{P}(A) \setminus \{\emptyset\}$.

(d) The subset $\{(A', B') \in \mathfrak{P}(A) \times \mathfrak{P}(A) \mid A' \subseteq B'\}$ is a relation on $\mathfrak{P}(A)$, called the inclusion relation.

(e) The subset $< := \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ is a relation, called the strict order relation on $\mathbb{N}$. Further, $\mathrm{Dom}(<) = \mathbb{N}$ and $\mathrm{Rng}(<) = \mathbb{N} \setminus \{0\}$. In particular, $< \subsetneq \mathrm{Dom}(<) \times \mathrm{Rng}(<)$.

(f) If R and S are relations form A to B and if $R \subseteq S$, then R is also a relation from A to B and $\mathrm{Dom}(R) \subseteq \mathrm{Dom}(S)$ and $\mathrm{Rng}(R) \subseteq \mathrm{Rng}(S)$. In particular, if R and S are relations from A to B, then $R \cap S$ is also relation from A to B and $\mathrm{Dom}(R \cap S) \subseteq \mathrm{Dom}(R) \cap \mathrm{Dom}(S)$ and $\mathrm{Rng}(R \cap S) \subseteq \mathrm{Rng}(R) \cap \mathrm{Rng}(S)$.

[1] More generally, for every positive integer n, one can define n-ary relation as a subset of $A^n := A \times \cdots \times A$ (n-times). We shall rarely consider n-ary relation for $n \neq 2$ and so by relation from now on we shall mean a binary relation unless otherwise specified.

T3.3 Let X and Y be sets.

(a) The map $\Gamma : \mathrm{Maps}(X,Y) \to \mathfrak{P}(X\times Y)$ defined by $f \mapsto \Gamma_f := \{(x,f(x)) \mid x\in X\}$ the graph of f is injective. (**Remark :** Therefore (if we identify maps with its graphs) every map from X to Y is a relation from X to Y. Further, since the map Γ is not surjective if $X\neq\emptyset$ and $(|X|,|Y|)\neq(1,1)$, in this case there are relations from X to Y which are not maps from X to Y. For example, each of the relations $\{(x,y),(x,y') \mid x\in X\ ;\ y,y'\in Y, y\neq y'\}$ and (if $|X|>1$) $\{(x,y)\mid x\in X, y\in Y\}$ from X to Y is not a map from X to Y.)

(b) The map $\mathfrak{P}(X\times Y)\to\mathfrak{P}(Y)^X$ defined by $R\mapsto \big(x\mapsto\{y\in Y\mid xRy\}\big)$ is bijective. What is the inverse of this map? (**Remark :** With this bijection, one can identify every relation $R\subseteq X\times Y$ between X and Y as a map from X into $\mathfrak{P}(Y)$.)

T3.4 Let A and B be sets.

(a) (Inverse relation) If R is a relation from A to B, then $R^{-1} := \{(y,x)\in B\times A \mid (x,y)\in R\}$ is a relation from B to A and is called the inverse of the relation R. (**Remarks :** For example, $(\Delta_A)^{-1}=\Delta_A$, $(<)^{-1}=\{(n,m)\in\mathbb{N}\times\mathbb{N}\mid n>m\}$ is the inverse relation of the strict order relation on $\mathbb{N}$ and $(A\times B)^{-1}=B\times A$. Even if a relation R from A to B is a map, i.e., $R=\Gamma_f$ for some $f\in\mathrm{Maps}(A,B)$, the inverse relation R^{-1} need not be a map from B to A. For example, the inverse relation $R_c^{-1}=\{(c,x)\mid x\in A\}$ of the constant relation $R_c:=\{(x,c)\mid x\in A\}$, $c\in B$ is not a map from B to A if either $|A|>1$ or $|B|>1$. Further, see the part (d) below.)

(b) (Composition of relations) Let R be a relation from A to B and let S be a relation from B to Z. We may define the composition of these relations by

$$S\circ R=\{(x,z)\in A\times Z \mid \text{there exists } y\in B \text{ such that } (x,y)\in R \text{ and } (y,z)\in S\}.$$

which is a relation from A to Z.

(c) If $R=\Gamma_f$ and $S=\Gamma_g$, then $S\circ R=\Gamma_{g\circ f}$. (**Remarks :** This mean that we have extended the definition of the composition from the set of maps to the set of relations. If R is a relation from A to B with $R^{-1}\circ R\subseteq\Delta_A$ and if for every $x\in A$, there exists $y\in B$ with $(x,y)\in R$, then R is a map from A to B.)

(d) (Associativity of composition) If furthermore T is a relation from Z to W, then

$$T\circ(S\circ R)=(T\circ S)\circ R.$$

(e) $(S\circ R)^{-1}=R^{-1}\circ S^{-1}$.

(f) (Product relations) Let R be a relation from A to B and let R' be a relation from A' to B. We may define the product of these relations by

$$R\times R' := \{((x,x'),(y,y'))\in(A\times A')\times(B\times B')\mid (x,y)\in R \text{ and } (x',y')\in R'\}.$$

which is a relation from $A\times A'$ to $B\times B'$.

T3.5 Definitions : Let A be a set. A relation $R\in\mathfrak{P}(A\times A)$ on A is called

(1) reflexive if aRa for all $a\in A$;

(2) symmetric if for $a,b\in A$, aRb implies bRa;

(3) transitive if for $a,b,c\in A$, aRb and bRc implies aRc;

(4) anti-symmetric if for $a,b\in A$, aRb and bRa implies $a=b$.

(5) (Equivalence relations) A relation R on A is called an equivalence relation if it is reflexive, symmetric and transitive. The identity relation Δ_A and the all-relation $A\times A$ on A are clearly equivalence relations on A.

T3.6 Let R be an equivalence relation on A. Then for $a\in A$, the subset $[a]_R=[a]=\{x\in A\mid (x,a)\in R\}$ is called the equivalence class of a under R (sometimes equivalence classes are also denoted by $\overline{a}$).

(1) For every $a\in A$, $a\in[a]$. In particular, $[a]\neq\emptyset$ for every $a\in A$ and $A=\bigcup_{a\in A}[a]$.

Proof. Immediate from (the reflexivity of R) $(a,a) \in R$ for every $a \in A$. •

(2) For all $a, b \in A$, the following statements are equivalent :

(i) $[a] = [b]$. (ii) $[a] \cap [b] \neq \emptyset$. (iii) $(a,b) \in R$.

Proof. For a proof we shall prove: (i)⇒(ii) : If $[a] = [b]$, then $a \in [a] = [b]$ and hence $a \in [a] \cap [b]$. (ii)⇒(iii) : Let $c \in [a] \cap [b]$, i. e. $(c,a) \in R$ and $(c,b) \in R$. Then from $(a,c) \in R$ (by the symmetry of R) and $(c,b) \in R$, it follows from (the transitivity of R) that $(a,b) \in R$. (iii)⇒(i) : Because of the symmetry of R, it is enough to prove that $[a] \subseteq [b]$. Let $c \in [a]$, i. e. $(c,a) \in R$. This together with $(a,b) \in R$, it follows that $(c,b) \in R$, i. e. $c \in [b]$. •

(3) (Quotient set of an equivalence relation) The set of equivalence classes in A under the relation R is denoted by A/R (read : "A modulo R") and is called the quotient set of A with respect to R.

(a) The canonical map $\pi : A \to A/R$, $x \mapsto [x]_R$ is clearly surjective and is called canonical projection of A onto A/R. The fibres of the canonical projection are precisely the equivalence classes (in A) under R, i. e. $\pi^{-1}([a]_R) = [a]_R$.

(b) An element $a \in A$ is called a representative of the equivalence class $[a]_R$; any other element $x \in A$ is a representative of $[a]_R$ if and only if $x \in [a]_R$ or equivalently $(x,a) \in R$.

(c) A (full or complete) representative system or a Fundamental domain for the quotient set A/R is a family x_i, $i \in I$ of elements in A such that the map $I \to A/R$ defined by $i \mapsto [x_i]$ is bijective, i. e., every equivalence class in A is represented by a unique element x_i, $i \in I$. In particular, a subset $A' \subseteq A$ is a representative system for A/R if and only if the restriction $\pi|A' : A' \to A/R$ of the canonical projection to A' is bijective.

(4) Let R and S be equivalence relations on the sets X and Y, respectively. Then the product relation $R \times S$ is an equivalence relation on $X \times Y$. What are the equivalence classes of the product relation $R \times S$? What is the quotient set $(X \times Y)/(R \times S)$?

T3.7 (Relation Matrix) Let $X := \{x_1, \ldots, x_m\}$, $Y := \{y_1, \ldots, y_m\}$ be finite sets and let R be a relation from X to Y. Then R can be specified by a matrix whose rows are labeled by the elements of X and whose columns are labeled by the elements of Y. In the i-th row and j-th column we write the entry 1 if $(x_i, y_j) \in R$ and 0 if $(x_i, y_j) \notin R$. This matrix is called a relation matrix of R and is usually denoted by $\mathfrak{A}(R)$.

(a) If $X = \{a,b\}$, $Y = \{c,d,e\}$ and $R = \{(a,c),(a,d),(b,e)\}$, $R' = \{(b,c),(b,d),(a,e)\}$. Then $\mathfrak{A}(R) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathfrak{A}(R') = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(b) Conversely, each $m \times n$ matrix $\mathfrak{A} = (a_{ij})$ of 0's and 1's defines a relation R from the set X to the set Y by the rule $(x_i, y_j) \in R$ if and only if $a_{ij} = 1$.

(c) Compute the matrices of the following relations :

(i) $=$ and $\leq$ on the sets $\{-1,0,1\}, \{-2,-1,0,1,2\}$.

(ii) $=$ and "negative of" on the sets $\{-1,0,1\}, \{-2,-1,0,1,2\}$.

(d) Show that the following statements are equivalent :

(i) R is both symmetric and anti-symmetric.

(ii) The matrix $\mathfrak{A}(R) = (a_{ij})$ is diagonal, that is, $a_{ij} = 0$ whenever $i \neq j$.

(iii) $R \subseteq \Delta_X$.

T3.8 Let $f : X \to Y$ be a map and let T be a relation on the set Y. Define a relation R on the set X by: $(x,x') \in R$ if and only if $(f(x), f(x')) \in T$. Prove that T is reflexive (respectively, symmetric, transitive), then R is also reflexive (respectively, symmetric, transitive). This generalize the Exercise 3.1 (how?). If the map f is bijective and if T is an equivalence relation, then R is also an equivalence relation on X. What is the relation between equivalence classes of R and those of T?

T3.9 Partitions of a set : Let A, be a set. A set of subsets of A is called a decomposition or a partition of A if the subsets are pairwise disjoint and if their union is whole A. More generally, an arbitrary family A_i, $i \in I$, of subsets of A is called a partition of A (parameterized by the index set I) if $A_i \cap A_j = \emptyset$ for all $i, j \in I$ with $i \neq j$ and if $\bigcup_{i\in I} A_i = A$. In this we shall also write $A = \biguplus_{i\in I} A_i$. [2] If $A = \bigcup_{i\in I} A_i$ without necessarily the condition of pairwise disjointness of A_i, $i \in I$, then the family A_i, $i \in I$, is called the covering of A.

The set of all partitions of A is denoted by $\mathfrak{Par}(A)$; this is a subset of the set $\mathfrak{P}(\mathfrak{P}(A))$. As usual for $n \in \mathbb{N}$, we put $\mathfrak{Par}_n(A) = \{\mathfrak{P} \in \mathfrak{Par}(A) \mid |\mathfrak{P}| = n\}$. Clearly the family $\mathfrak{Par}_n(A)$, $n \in \mathbb{N}$ is pairwise disjoint and $\cup_{n\in\mathbb{N}} \mathfrak{Par}_n(A) = \mathfrak{Par}(A)$.

(a) *The partitions A_i, $i \in I$, of A corresponds to the maps $f : A \to I$* : The decomposition $A_i, i \in I$, defines the map f by $f(a) := i$, if $a \in A_i$, and conversely, the map f defines the decomposition $A_i := f^{-1}(i)$, $i \in I$, of A. The subsets $A_i \neq \emptyset$ for all $i \in I$ if and only if f is surjective. If A is a finite set, then clearly every partition $\mathfrak{P}$ of A is also a finite set (and $|\mathfrak{P}| \leq |X|$).

(b) Show that the map $\alpha : \mathfrak{P}(X \times X) \to \mathfrak{P}(\mathfrak{P}(X))$, $R \mapsto \{\{y \in A \mid xRy\} \mid x \in X\}$ maps the subset $\mathfrak{Eq}(X) \subseteq \mathfrak{P}(X \times X)$ of all equivalence relations on X bijectively onto the set $\mathfrak{Par}(X)$ of all partitions of X. (**Hint :** This means to each equivalence relation R on X, α associates a unique partition $\alpha(R)$ of X and conversely. The partition corresponding to the equivalence relation R on X is denoted by $\mathfrak{p}_R$ and the equivalence relation corresponding to the partition $\mathfrak{p}$ of X is denoted by $R_\mathfrak{p}$. Then the maps $\mathfrak{P}(X) \to \mathfrak{Eq}(X)$, $\mathfrak{p} \mapsto \mathfrak{p}_R$ and $\mathfrak{Eq}(X) \to \mathfrak{Par}(X)$, $R \mapsto R_\mathfrak{p}$ are bijective and are inverses of each other. Moreover, if $\mathfrak{Eq}_r(X)$ is the set of all equivalence relations on X with exactly r equivalence classes. Then $|\mathfrak{Eq}_r(X)| = |\mathfrak{Par}_r(X)|$ and $\mathfrak{Eq}(X) = \biguplus_{r=0}^{n} \mathfrak{Eq}_r(X)$.)

T3.10 (Congruence relations) Let $n \in \mathbb{N}^+$ be a positive natural number. Two integers a and b are called congruent modulo n, if the difference $b - a$ is divisible by n. In this we write $a \equiv b \pmod n$ or $a \equiv_n b$.

The congruence modulo n $\equiv_n$ is equivalence relation on the set of integers $\mathbb{Z}$. Two integers are congruent modulo n if and only if their remainders (between 0 and $n-1$) after the division by n are equal. Therefore the numbers $0, \ldots, n-1$ form a full representative system for the quotient set $\mathbb{Z}/\equiv_n$; there are exactly n equivalence classes, namely,

$$\bar{i} := i + \mathbb{Z}n = \{i + kn \mid k \in \mathbb{Z}\}, \quad i = 0, \ldots, n-1;$$

these are also called the residue classes modulo n.

T3.11 Let $\mathfrak{p}$ and $\mathfrak{q}$ be two partitions of a set X. We say that $\mathfrak{p}$ is coarser than $\mathfrak{q}$ (or $\mathfrak{q}$ is finer than $\mathfrak{p}$) if $\mathfrak{p} \subseteq \mathfrak{q}$.

(a) Let R and S be two equivalence relations on a set X and let $\mathfrak{p}_R$ and $\mathfrak{p}_S$ be the corresponding (see T.37-(b)) partitions of X. Then S is stronger than R, i. e. $S \subseteq R$ if and only if $\mathfrak{p}_R$ is finer than $\mathfrak{p}_S$, i. e. $\mathfrak{p}_S \subseteq \mathfrak{p}_R$.

(b) If the relation R is induced by some map on X, then the result in part (a) is often expressed using maps, for this first let us define:

Let S be an equivalence relation on a set X and let $f : X \to Y$ be a map. We say that f is compatible with S if for all $x, x' \in X$, $(x, x') \in S$ implies $f(x) = f(x')$. For examples see part (c) below. With this definition, we can now reformulate the result in part (a) as follows:

Let S be an equivalence relation on a set X and let $f : X \to Y$ be a map. Then the following statements are equivalent:

[2] We don't insists — like some other authors do this — that all $A_i \neq \emptyset$.

(i) S is stronger than Γ_f.

(ii) The map f is compatible with S.

(iii) There exists a map $g : X/S \to Y$ such that $g \circ \pi_S = f$ (in this case one also say that f factors through p). Moreover, such a g is uniquely determined by f. (**Remark:** In this case we also say that the following diagram of maps is *commutative*:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_S"'] & Y \\ \overline{X} = X/S \arrow[ur, "g"'] & \end{tikzcd}$$
)

(c) Examples: Let X be the set of all persons in a country IND and let S be the relation: $(x,y) \in S$ if and only if x and y live in the same district. Further, let $f : X \to Y :=$ the set of all states in IND be defined by $f(x) :=$ the state the person x lives in. Then f is compatible with the relation S because two persons living in the same district obviously live in the same state. But if we define $f : X \to \mathbb{R}$ by $f(x) :=$ the height of x, then f is not compatible with S (except in the unlikely event that all persons in each district are of equal height).

(d) Let R and S be equivalence relations on the sets X and Y, respectively and let $f : X \to Y$ be a map which is compatible with R and S, i. e. for all $x, x' \in X$, $(x,x') \in R$ implies that $(f(x), f(x')) \in S$. Prove that there exists a unique map $g : X/R \to Y/S$ such that $g \circ \pi_R = g \circ \pi_S$, i. e. the diagram of maps

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_R"'] & Y \arrow[d, "\pi_S"] \\ X/R \arrow[r, "g"] & Y/S \end{tikzcd}$$

is commutative.

(e) Examples: Let $m,n \in \mathbb{N}^+$ be two non-zero natural numbers. For an integer $a \in \mathbb{Z}$, let $\lambda_a : \mathbb{Z} \to \mathbb{Z}$ and $\tau_a : \mathbb{Z} \to \mathbb{Z}$ be the left multiplication $x \mapsto ax$ and the translation $x \mapsto x + a$, $x \in \mathbb{Z}$, maps by a, respectively. Determine the conditions on integers a, m, n so that

(i) λ_a is compatible with the congruence relation $\equiv_m$.

(ii) τ_a is compatible with the congruence relation $\equiv_m$.

(iii) λ_a is compatible with the congruence relations $\equiv_m$ and $\equiv_n$.

(iv) τ_a is compatible with the congruence relations $\equiv_m$ and $\equiv_n$.

Moreover, in the cases of compatibility describe the unique maps (see parts (b) and (d)) induced by the maps λ_a and τ_a.

(f) Which of the following maps are compatible with the equivalence relation R defined in the Exercise 3.6:

(i) $\mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, $(m,n) \mapsto m + n$.

(ii) $\mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, $(m,n) \mapsto m \cdot n$.

(iii) $\mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, $(m,n) \mapsto m - n$.