E0 221 Discrete Structures / August-December 2013

(ME, MSc, Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 10:00–11:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Quizzes : During Wednesday, Lectures on **Aug 28; Sept 18; (Monday) Oct 14; Oct 30;** **Time :** 10:00 -10:15
1-st Midterm : Saturday, September 14, 2013; 14:00 -16:30 **2-nd Midterm :** Saturday, October 12, 2013; 10:00 -12:00
Final Examination : Saturday, December 07, 2013, 14:00 -17:00

Evaluation Weightage : Quizzes (Four) + Midterms (Two) : 50% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

4. Orders

- **Solution of the $*$-Exercise 4.10 carries 10 Bonus Points.**

4.1 On the set $\mathbb{N}^+$ of positive natural numbers, let $|$ denote the relation "divides", i. e. for $m,n \in \mathbb{N}^+$, $m \mid n$ if and only if $n = am$ for some $a \in \mathbb{N}^+$. Show that:

(a) $|$ is an order on $\mathbb{N}^+$ and that the element 1 is the least element.

(b) The prime numbers are precisely the minimal elements in $(\mathbb{N}^+ \setminus \{1\}, |)$.

(c) Draw the Hasse-Diagrams for the set of divisors of 12 and 30.

(d) The chains in $(\mathbb{N}^+, |)$ are either finite or an infinite sequence of the type

$$C = (q_0,\ q_0q_1,\ q_0q_1q_2,\ \dots\ )$$

with $q_n \in \mathbb{N}^+$, $q_n \geq 2$ for all $n \geq 1$. Moreover, C is a maximal chain if and only if the sequence $(q_0,\ q_1,\ q_2,\ \dots\ )$ is infinite and $q_0 = 1$ and q_n, $n \geq 2$ are all prime numbers. (**Remark :** For an ordered set $(X, \leq)$, the set $\mathscr{C}(X)$ of chains in X is an ordered set with the natural inclusion. A chain $C \in \mathscr{C}(X)$ is called a m a x i m a l c h a i n in X if it is a maximal element in the ordered set $(\mathscr{C}(X), \subseteq)$.)

4.2 On the lower half-plane $\mathrm{H}_{\leq 0} := \{(x,y) \in \mathbb{R}^2 \mid y \leq 0\} \subseteq \mathbb{R}^2$, define a relation $\preceq$ by : $(x_1,y_1) \preceq (x_1,y_2)$ if $x_1 = x_2$ and $y_1 \preceq y_2$. Show that $\preceq$ is an order on $\mathrm{H}_{\leq 0}$. Determine its maximal elements.

4.3 **(a)** Let I be any set and let $\mathfrak{P}(I)$ be the power set of I. The natural inclusion $\subseteq$ defines and order on $\mathfrak{P}(I)$. Moreover, the ordered set $(\mathfrak{P}(I), \subseteq)$ is a complete ordered set.

(b) The ordered set $(\mathbb{N}, \leq)$ of natural numbers $\mathbb{N}$ with the usual order $\leq$ is usually denoted by ω. Show that ω is a conditionally complete ordered set and all of its finite subsets are complete ordered. In particular, for each $n \in \mathbb{N}$, the ordered set $\Delta_n := (\{0,1,\dots,n\}, \leq)$ is a complete ordered set. What are the lower cuts in ω and in Δ_n?

(c) Give an order $\preceq$ on the set $\mathbb{N}$ of natural numbers so that $(\mathbb{N}, \preceq)$ is a complete ordered set.

4.4 Give an order $\preceq$ on the set $\mathbb{N} \times \mathbb{N}$ so that:

(a) The ordered set $(\mathbb{N} \times \mathbb{N}, \preceq)$ has infinitely many lower cuts.

(b) The ordered set $(\mathbb{N} \times \mathbb{N}, \preceq)$ has no lower cuts.

4.5 Let $(X, \leq)$ be an dense ordered set and let a and b be two elements of X with $a < b$. Prove that there exist infinitely elements distinct elements $x \in X$ such that $a < x < b$, i. e. the open interval (a,b) of X is infinite.

4.6 Let $(X, \leq)$ be a simply ordered set.

(a) Show that X is dense if and only if no element of X has an immediate successor.

(b) Let $x,y \in X$ and $I(x)$ be an initial segment of X. Then show that:

(i) If the initial segment $I(y)$ of X is a subset of $I(x)$, then $I(y)$ is also an initial segment of $I(x)$.

(ii) If $I(y)$ is an initial segment of $I(x)$, then $I(y)$ is also an initial segment of X. (Recall that for every element a in an arbitrary ordered set $(X,\leq)$, the set $I_X(a) := I(a) := \{x \in X \mid x < a\}$ is called the initial segment of X determined by a. If $I(a) \neq \emptyset$, then $I(a)$ is called a proper initial segment of X.)

(c) Give an example of a simply ordered set $(X,\leq)$ such that every initial segment of which contains only finitely many elements.

4.7 **(a)** Well order the set of natural numbers $\mathbb{N}$ so that exactly five of its elements do not have predecessors.

(b) Give three well orders on the set $\mathbb{Z}$ of integers.

(c) Give a well order on the set $\mathbb{Q}$ of rational numbers. (**Hint :** Use the additional fact that $\mathbb{Q}$ is *countable*, i. e. there exists a bijective map $q : \mathbb{N} \to \mathbb{Q}$. With this use T4.26-(f).)

4.8 Let X and Y be well ordered sets. Let $\leq$ defined by:

For $x_1,x_2 \in X$ and $y_1,y_2 \in Y$: $(x_1,y_1) \leq (x_2,y_2)$ if and only if $y_1 < y_2$ or $y_1 = y_2$ and $x_1 \leq x_2$.

(a) Show that $\leq$ is a well order on $X \times Y$.

(b) Is the *dictionary order* on $X \times Y$ a well ordering?

4.9 Let $(X,\leq_X)$ and $(Y,\leq_Y)$ be ordered sets. A map $f : X \to Y$ is said to be order-preserving (or order-homomorphism) if for all $x,x' \in X$ with $x \leq x'$ implies that $f(x) \leq f(x')$. (**Remark :** When there is no room for confusion, we shall drop (but remember them at appropriate places) the suffixes in the notations of the orders $\leq_X$ and $\leq_Y$ on X and Y, respectively. With this an order-homomorphism is also called an increasing map. Similarly, if for all $x,x' \in X$ with $x < x'$ implies that $f(x) < f(x')$, then we say that f is strictly increasing. A map is strictly increasing if and only if it is injective order-homomorphism.)

(a) The successor map $\sigma : \omega = (\mathbb{N},\leq) \to \omega = (\mathbb{N},\leq)$, $n \mapsto n+1$, is an order-preserving. The map $\lambda_2 : \omega \to \{2n \mid n \in \mathbb{N}\}$, $n \mapsto 2n$, is also order-preserving.

(b) The bijection $\sigma : (\mathbb{N},\leq) \to \mathbb{N} \setminus \{0\}$, $n \mapsto n+1$, is an order-isomorphism. (**Remark :** A bijective map $f : X \to Y$ of ordered sets is called an order-isomorphism if both f and its inverse f^{-1} are order-homomorphisms.)

(c) If $(X,\leq_X)$ is simply ordered and if $f : (X,\leq_X) \to (Y,\leq_Y)$ is an injective order homomorphism, then f is an order-isomorphism onto its image $f(X)$. Give an example of a bijective order-homomorphism which is not an order-isomorphism.

(d) Let X be an arbitrary set and let $\varphi : (\mathfrak{P}(X),\subseteq) \to (\mathrm{Maps}\,(X,\{0,1\}),\leq)$ be the map defined by $A \mapsto \chi_A$. Then φ is an order-isomorphism. (**Remark :** Note that the order $\leq$ on $\mathrm{Maps}\,(X,\{0,1\})$ is defined pointwise, i. e. $f \leq g$ if and only if $f(x) \leq g(x)$ for $x \in X$.)

***4.10** Let $(X,\leq)$ be an ordered set in which every subset has a least upper bound and has a greatest lower bound. Further, let $f : X \to X$ be an increasing map and let $\mathrm{Fix}_X(f) := \{x \in X \mid f(x) = x\}$ be the fixed points of f. Show that:

(a) If $A := \{x \in X \mid f(x) < x\} \neq \emptyset$ and if a is its greatest lower bound, then either $a \in \mathrm{Fix}_X(f)$ or $f(a) \in \mathrm{Fix}_X(f)$. (**Hint :** Since $a \leq x$ for all $x \in A$, it follows that $f(a) \leq f(x) < x$ for all $x \in A$. If $f(a) \neq a$, then $f(a) < a$, i. e. $a \in A$ and $a = \mathrm{Min}\,(A)$ and hence $f(a) \notin A$. This proves that $f(f(a)) = f(a)$.)

(b) If $\{x \in X \mid x < f(x)\} \neq \emptyset$ and if b is its least upper bound, then either $b \in \mathrm{Fix}_X(f)$ or $f(b) \in \mathrm{Fix}_X(f)$. (**Hint :** Similar argument as in part (a).)

(c) $\mathrm{Fix}_X(f) \neq \emptyset$. (**Hint :** immediate from parts (a) and (b).)

(d) The least upper bound and the greatest lower bound of $\mathrm{Fix}_X(f)$ belong to $\mathrm{Fix}_X(f)$.

Below one can see Lecture Notes.

Lecture Notes

Relations which *order* a set occur in all domains of mathematics and in branches of the empirical sciences. Many fundamental questions in mathematics are directly concerned with the notion of an order. Therefore, the study of the general properties of ordered sets is of capital importance. There are almost an endless number of interesting theorems about various ordering relations and their properties.

T4.1 (Orders — Simple orders, Chains) Let X be a set and let $\leq$ be a relation on X. Then $\leq$ is called an order (relation) on X if :

(i) For all $x \in X$, $x \leq x$. (Reflexivity)

(ii) For all $x, y \in X$, from $x \leq y$ and $y \leq x$, it follows that $x = y$. (Antisymmetry)

(iii) For all $x, y, z \in X$, from $x \leq y$ and $y \leq z$, it follows that $x \leq z$. (Transitivity)

Moreover, if $\leq$ satisfies the condition:

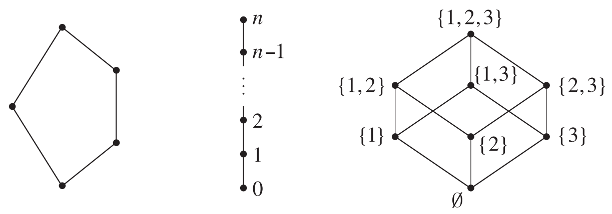
(iv) Every two elements $x, y \in X$ are comparable, i. e. either $x \leq y$ or $y \leq x$.

then $\leq$ is called a simple (or total, or linear) order on X. A set X with an order $\leq$ is denoted by a pair $(X, \leq)$ and is called an ordered set.

Let $(X, \leq)$ be an ordered set. If $Y \subseteq X$, then the relation induced by $\leq$ is a order on Y, denoted again by $\leq |_Y$. A totally ordered subset of an ordered set $(X, \leq)$ is also called a chain in X.

For $x, y \in X$, we also write $x < y$ for "$x \leq y$ and $x \neq y$". Further, for $x \leq y$ (respectively, $x < y$), we also write $y \geq x$ (respectively, $y > x$) and the so defined order $\geq$ is called the opposite order of $\leq$ which is also denoted by $\leq^{\mathrm{op}}$.

(1) (Hasse-Diagram) An illustration of an order can be given by using a directed graph in which the vertices are the points of the plane and in general use the following simplification: Note that the arrows are from bottom to top and hence they can be omitted. Moreover, all loops and all the connecting edges are opened-up on the basis of the transitivity of the order relation.

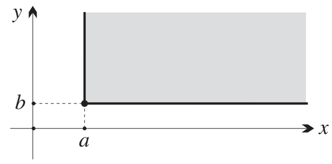


Such a diagram for an order relation is called a Hasse-Diagram. A typical example for a Hasse-Diagram is the left-figure above. The other both Hasse-Diagrams are for the natural order on the set $\{0, 1, \ldots, n\}$ respectively, for the inclusion $\subseteq$ on the power-set $\mathfrak{P}(\{1, 2, 3\})$.

(2) (Product-Order and Lexicographic order) Let $(X_1, \leq_1), \ldots, (X_n, \leq_n)$ be ordered sets. On the cartesian product $X_1 \times \cdots \times X_n$, we define $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if and only if $x_i \leq_i y_i$ for all $i = 1, \ldots, n$. Clearly, $\leq$ is an order on $X_1 \times \cdots \times X_n$. This order is called the product order on $X_1 \times \cdots \times X_n$.

On the cartesian product the relation $\leq_{\mathrm{lex}}$ defined by $(x_1, \ldots, x_n) \leq_{\mathrm{lex}} (y_1, \ldots, y_n)$ if and only if either $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ or $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$ and for the smallest index i with $x_i \neq y_i$, $x_i <_i y_i$. Clearly, $\leq_{\mathrm{lex}}$ is an order on $X_1 \times \cdots \times X_n$. This order is called the lexicographic (or dictionary) order on $X_1 \times \cdots \times X_n$.

In the lexicographic order $(1, 2) \leq_{\mathrm{lex}} (2, 1)$ (in the lexicon the words starting with "ab" come before the words starting with "ba"). But in the product order $(1, 2)$ and $(2, 1)$ are not comparable. If all $(X_i, \leq_i)$, $i = 1, \ldots, n$, are simply ordered, then so is $(X_1 \times \cdots \times X_n, \leq_{\mathrm{lex}})$.



$\{(x,y) \mid (a,b) \leq (x,y)\}$ if $\leq$ is the product order on $\mathbb{R} \times \mathbb{R}$

$\{(x,y) \mid (a,b) \leq_{\text{lex}} (x,y)\}$ if $\leq_{\text{lex}}$ is the lexicographic order on $\mathbb{R} \times \mathbb{R}$

T4.2 (Distinguished elements and subsets) Let $(X, \leq)$ be an ordered set.

(a) An element $b \in X$ is called an immediate successor of $a \in X$ if $a < b$ and there is no element $c \in X$ such that $a < c < b$. If b is an immediate successor of a, then a is called an immediate predecessor of b.

(b) An element $a \in X$ is called a minimal element of X if $x \leq a$ implies $x = a$ for every $x \in X$. In other words, a minimal element of X if no element of X precedes a.

(c) An element $b \in X$ is called a maximal element of X if $b \leq x$ implies $x = b$ for every $x \in X$. In other words, b maximal element of X if no element of X exceeds b.

Ordered sets may or may not have minimal or maximal elements. Even if minimal (respectively, maximal) elements exist they may or may not be unique, see Examples in T4.4.

(d) An element $a \in X$ is called a least or smallest or first or minimum of X if $x \leq a$ for every $x \in X$. Observe that if there exists a mimimum element a in X, then it is unique. Indeed, if $a' \in X$ is such that $a' \leq x$ for every $x \in X$, then on one hand $a \leq a'$ and on the other hand $a' \leq a$ and hence $a' = a$ by the anti-symmetry of $\leq$. Therefore we are justifies to call a minimum element in X as *the* minimum of X and is denoted by $\operatorname{Min}(X)$.

(e) An element $b \in X$ is called a greatest or largest or last or maximum of X if $b \leq x$ for every $x \in X$. Observe that if there exists a maximum element b in X, then it is unique. Indeed, if $b' \in X$ is such that $x \leq b'$ for every $x \in X$, then on one hand $b' \leq b$ and on the other hand $b \leq b'$ and hence $b' = b$ by the anti-symmetry of $\leq$. Therefore we are justifies to call a maximum element in X as *the* maximum of X and is denoted by $\operatorname{Max}(X)$.

The minimum and the maximum elements in an ordered set $(X, \leq)$ are comparable with every element of X. However, they may or may not exists, see Examples in T4.4.

(f) Let $A \subseteq X$. An element $x \in X$ is called a lower bound (respectively, an upper bound) for A if $x \leq a$ (respectively, $a \leq x$) for every $a \in A$. Note that it is not required that a lower bound or an upper bound of a subset A of an ordered set $(X, \leq)$ belong to A, see Examples in T4.4. The set of all lower bounds (respectively, upper bounds) of A in X is denoted by $\mathrm{LB}_X(A)$ (respectively, $\mathrm{UB}_X(A)$). A subset A is called bounded below (respectively, bounded above) if A has a lower bound (respectively, upper bound) in X, i. e. if $\mathrm{LB}_X(A) \neq \emptyset$ (respectively, $\mathrm{UB}_X(A) \neq \emptyset$).

(g) Let $A \subseteq X$. The greatest (respectively, the least) element of the set $\mathrm{UB}_X(A)$ (respectively, $\mathrm{LB}_X(A)$) of all upper bounds (respectively, lower bounds) of A is called *the* greatest lower bound or the supremum (respectively, *the* least upper bound or the infimum) of A (in X) and is denoted by $\mathrm{GLB}_X(A)$ or $\mathrm{Sup}_X(A)$ (respectively, $\mathrm{LUB}_X(A)$ or $\mathrm{Inf}_X(A)$. Note that if $\mathrm{GLB}_X(A)$ (respectively, $\mathrm{LUB}_X(A)$ exists, then it is unique. However, $\mathrm{GLB}_X(A)$ (respectively, $\mathrm{LUB}_X(A)$ may or may not exists, see Examples in T4.4.

Note that in any ordered set $(X, \leq)$, we have $\operatorname{Min}(X) = \mathrm{LUB}_X(\emptyset) = \mathrm{Inf}(\emptyset)$ and $\operatorname{Max}(X) = \mathrm{GLB}_X(\emptyset) = \mathrm{Sup}(\emptyset)$.

T4.3 Let $\mathbb{N}$ be the set of natural numbers. Show that each of the relations $\preceq$ defined below on the set $\mathbb{N}$ of natural numbers are orders: For $m, n \in \mathbb{N}$, define $m \preceq n$ if

(a) $m \leq n$ (in the usual sense) and m and n have the same parity.

(b) m is even, and m and n have different parity.

(c) $m = n$ or upon the division of m and n by 5, m yields the smaller remainder.

(d) $m = n$ or in the division in (c), m and n yield the same remainder and $m < n$.

T4.4 Notation and Examples : For the sake of convenience in giving many examples , we shall use the following configuration such as:

$$\{\ \dots\ ,\ (\ \dots\ ;a;b;\ \dots\ ;c;\ \dots\ )\ ,\ (\ \dots\ ;m;n;\ \dots\ )\ ,\dots\ \}$$

to represent the ordered set $(X,\leq)$, where $X = \{\ \dots\ ,a\,,b\,,c\,,\ \dots\ ,\ ,m\,,n\,,\ \dots\ \}$ and where $x \leq y$ if and only if x and y are contained in the same parentheses and x is not written to the right of y.

(a) $A = \{(a)\}$ (exactly one minimal element and one maximal element, $\mathrm{Min}A = \mathrm{Max}A = a$).

(b) $B = \{(a;b),(c;d)\}$ (two minimal elements $\{a,c\}$ and two maximal elements $\{b,d\}$).

(c) $C = \{\,(\dots\,;a_2\,;a_1\,;b_1\,;b_2\,;\,\dots)\}$ (no minimal elements and no maximal elements).

(d) $D = \{\,(\dots\,;a_2\,;a_1\,;\ \dots\ ;b_2\,;b_1\,;c_1\,;c_2\,;\ \dots\ ;d_2\,;d_1\,)\,,(\,a\,;b\,;c\,)\}$ (exactly one minimal element a, no minimum and two maximal elements $\{d_1,c\}$, no maximum.).

(e) $E = \{\,(\,a\,;b\,)\,,(\,c;\ \dots\ ;d_2\,;d_1\,;e_1\,;e_2;\ \dots\ ;h\,)\,,(\ \dots\ ;m_2\,;m_1\,)\}$ (two minimal elements $\{a,c\}$ and three maximal elements $\{b,h,m_1\}$).

(f) $F = \{\,(\ \dots\ ;a_2\,;a_1\,;b_1\,;b_2;\ \dots\ )\,,(\ \dots\ c_2\,;c_1;e_1\,;e_2\,;\dots\ )\}$ (no minimal elements and no maximal elements).

(g) $G = \{\,(a_1\,;a_2\,;a_3\,;\ \dots\ ;b_3\,;b_2\,;b_1\,)\}$ (the minimum a_1 and the maximum b_1).

(h) The set of natural numbers (respectively of integers) with the usual order $\leq$ is represented as:

$$(\mathbb{N},\leq) = \{\ (0;1;2;\ \dots\ )\} \quad \text{(respectively} \quad (\mathbb{Z},\leq) = \{\ (\ \dots\ ;-2;-1;0;1;2;\ \dots\ )\}\text{)}.$$

(**Remark :** Such a representation for the set $\mathbb{Q}$ of rational numbers is more complicated and that for the set $\mathbb{R}$ of real numbers is not possible(?).)

T4.5 Give an example of an ordered set $(X,\leq)$ such that:

(a) There are exactly three minimal elements and two maximal elements and neither a minimum nor a maximum.

(b) Every non-empty subset Y of X has a least upper bound, but not necessarily a lower bound.

(c) Every non-empty bounded above subset has a least upper bound, but not every subset has a lower bound.

T4.6 Let $(X,\leq)$ be an ordered set.

(1) (D u a l) The inverse (or opposite) relation $\leq^{-1}$ (or $\leq^{\mathrm{op}}$) on X is again an order on X. Moreover $(\leq^{-1})^{-1} = (\leq^{\mathrm{op}})^{\mathrm{op}} = \leq$. The ordered set $(X,\leq)^{\mathrm{op}} := (X,\leq^{\mathrm{op}})$ is called the d u a l (or o p p o s i t e) of $(X,\leq)$. Clearly, the ordered sets $(X,\leq)$ and $(X\ \leq^{\mathrm{op}})$ are duals of each other.

(2) (D u a l s o f D i s t i n g u i s h e d e l e m e n t s a n d s u b s e t s) It is clear that a d i s - t i n g u i s h e d element – a *minimal*, a *maximal*, the *least* and the *greatest* element in $(X,\leq)$ becomes its c o u n t e r p a r t (or d u a l) – a *maximal*, a *minimal*, the *greatest* and the *least* element in the dual ordered set $(X,\leq^{\mathrm{op}})$. Moreover, a *lower bound*, the *subset* $\mathrm{LB}_X(Y)$ *of lower bounds*, an *upper bound*, the *subset* $\mathrm{UB}_X(Y)$ *of upper bounds*, the *greatest lower bound* $\mathrm{GLB}_X(Y)$ and the *least upper bound* $\mathrm{LUB}_X(Y)$ of a subset Y of X in $(X,\leq)$ becomes its d u a l – an *upper bound*, the *subset* $\mathrm{UB}_{X^{\mathrm{op}}}(Y)$ *of upper bounds*, a *lower bound*, the *subset* $\mathrm{LB}_{X^{\mathrm{op}}}(Y)$ *of lower bounds*, the *least upper bound* $\mathrm{GLB}_{X^{\mathrm{op}}}(Y)$ and the *greatest lower bound* $\mathrm{LUB}_{X^{\mathrm{op}}}(Y)$ of the subset Y of X in the dual ordered set $(X,\leq^{\mathrm{op}})$.

(3) For a statement $\mathscr{S}$ in an ordered set $(X,\leq)$, let $\mathscr{S}^{\mathrm{dual}}$ be the statement (called the dual of $\mathscr{S}$) obtained from the statement $\mathscr{S}$ by changing every distinguished elements and subsets to their duals and $\leq$ to $\leq^{\mathrm{op}}$.

(Duality Theorem) *A statement $\mathscr{T}$ is a theorem in every ordered set if and only if its dual $\mathscr{T}^{\text{dual}}$ is also a theorem in every ordered set.* (**Proof** Since the distinguished elements of X as well as those of a subset Y of X in every ordered set $(X,\leq)$ become their duals in the dual ordered set $(X,\leq^{\text{op}})$ and the ordered sets $(X,\leq^{\text{op}})$ and $(X,\leq)$ are duals of each other, the assertion is immediate.)

T4.7 (Dual Theorems) We give some concrete examples of the dual theorems.

(1) $\mathscr{T}$: *An ordered set $(X,\leq)$ has a least element if and only if the empty-set $\emptyset$ has a least upper bound in X.* (**Remark :** Note that $\mathscr{T}$ is a theorem in every ordered set $(X,\leq)$: For a proof note that $\text{UB}_X(\emptyset)=X$ and hence $\text{Min}(X)=\text{LUB}_X(\emptyset)=\text{Min}(\text{UB}_X(\emptyset))$.)

The dual of $\mathscr{T}$ is the following:

$\mathscr{T}^{\text{dual}}$: *An ordered set $(X,\leq)$ has a greatest element if and only if the empty-set $\emptyset$ has a greatest lower bound in X.*

(2) $\mathscr{T}$: *Let $(X,\leq)$ be an ordered set and let $Y\subseteq X$ and $z:=\text{LUB}_X(\text{LB}_X(Y))$. Then*

$$\text{LUB}_X(\text{LB}_X(Y))\in\text{LB}_X(Y) \quad \textit{and} \quad \text{LUB}_X(\text{LB}_X(Y))=\text{GLB}_X(Y)\,.$$

(**Remark :** Note that $\mathscr{T}$ is a theorem in every ordered set $(X,\leq)$: For a proof note that $z\leq y$ for every $y\in Y$ and hence $z\in\text{LB}_X(Y)$. On the other hand $x\leq z$ for every $x\in\text{LB}_X(Y)$. Therefore $z=\text{GLB}_X(Y)$.)

The dual of $\mathscr{T}$ is the following:

$\mathscr{T}^{\text{dual}}$: *Let $(X,\leq)$ be an ordered set and let $Y\subseteq X$. Then*

$$\text{GLB}_X(\text{UB}_X(Y))\in\text{UB}_X(Y) \quad \textit{and} \quad \text{GLB}_X(\text{UB}_X(Y))=\text{LUB}_X(Y)\,.$$

T4.8 Theorem : *Let $(X,\leq)$ be an ordered set. The following two statements are equivalent* :

(i) $\text{LUB}_X(Z)$ *exists for every non-empty bounded above subset Z of X.*

(ii) $\text{GLB}_X(Y)$ *exists for every non-empty bounded below subset Y of X.*

(**Proof** (i)⇒(ii): Let Y be a non-empty bounded below subset of X. Then $\text{LB}_X(Y)\neq\emptyset$ and is bounded above by every element of Y. Therefore $\text{LUB}_X(\text{LB}_X(Y))$ exists by (i) and by $\text{LUB}_X(\text{LB}_X(Y))=\text{GLB}_X(Y)$. The converse (implication (ii)⇒(i)) follows from the duality Theorem T4.6 (3).)

As a corollary of the above Theorem we have:

T4.9 Corollary : *Let $(X,\leq)$ be an ordered set. The following two statements are equivalent* :

(i) $\text{LUB}_X(Z)$ *exists for every subset Z of X.* (ii) $\text{GLB}_X(Y)$ *exists for every subset Y of X.*

T4.10 (Complete and Conditionally Complete Ordered Sets) Let $(X,\leq)$ be an ordered set. We say that X is complete if every subset of X has a least upper bound (or equivalently, every subset has a greatest lower bound).

We say that X is conditionally complete if every non-empty subset of X that is bounded above has a least upper bound (or equivalently, every non-empty subset that is bounded below has a greatest lower bound).

Clearly, a complete ordered set has the greatest ($\text{Inf}\,\emptyset$) and the least ($\text{Sup}\,\emptyset$) element. Further, every complete ordered set is conditionally complete.

T4.11 (Lower Cuts) Let $(X,\leq)$ be an Ordered Set. A subset L of X is called a lower cut of X if

(i) $\emptyset\neq L\subsetneq X$, i. e. L is a non-empty proper subset of X.

(ii) L has no greatest element, i. e. $\text{Max}\,L$ does not exist.

(iii) If $x\in L$, then every element $y\in X$ with $y\leq x$ is also an element of L.

With this definition we have the following:

T4.12 Theorem : *Let $\mathscr{L}(X)$ be the set of lower cuts of an ordered set $(X,\leq)$. Then the ordered set $(\mathscr{L}(X),\subseteq)$ is conditionally complete. Moreover, if $(X,\leq)$ is simply ordered, then $(\mathscr{L}(X),\subseteq)$ is also simply ordered.*
(**Proof :**

•)

T4.13 Theorem : *Let $(X,\leq)$ be a conditionally complete ordered set and $f: X\to X$ be a increasing map from X to X. If there are $a,b\in X$ such that $a\leq f(a)\leq f(b)\leq b$, then there exists an element $c\in X$ such that $a\leq c\leq b$ and $f(c)=c$. In particular, f has a fixed point.*

(**Proof :** Let $Y:=\{y\in X\mid a\leq y\leq b \text{ and } y\leq f(y)\}$. Then $a\in Y$ and b is an upper bound for Y. Therefore by assumption $c:=\mathrm{LUB}_X(Y)$ exists and $a\leq c\leq b$. We shall prove $f(c)=c$. Since $y\leq c$ for every $y\in Y$ and f is increasing, we have $y\leq f(y)\leq f(c)$ for every $y\in Y$. Therefore $f(c)$ is an upper bound for Y and hence $c\leq f(c)$. Further, $f(c)\leq f(f(c))$. On the other hand, since f is increasing, $a\leq f(a)\leq f(c)\leq f(b)\leq b$. Therefore $f(c)\in Y$ and $f(c)\leq c$, since $c=\mathrm{LUB}_X(Y)$. This proves that $f(c)=c$. •)

T4.14 Corollary : *Let $(X,\leq)$ be a complete ordered set and $f: X\to X$ be a increasing map from X to X. Then f has at least one fixed point.* (**Proof :** Since X is complete, it has a least and greatest elements, put $a:=\mathrm{Min}X$ and $b:=\mathrm{Max}X$. Then $a\leq f(a)\leq f(b)\leq b$ and hence we can apply Theorem T4.13. **Variant :** See Exercise 4.10. •)

T4.15 (Initial Segments) Let $(X,\leq)$ be an ordered set. For every element $a\in X$, the set $I(a):=\{x\in X\mid x<a\}$ is called the initial segment of X determined by a. If $I(a)\neq\emptyset$, then $I(a)$ is called a proper initial segment of X.

With this definition we have the following:

T4.16 Theorem : *Let $\mathscr{I}(X)$ be the set of all initial segments of an ordered set $(X,\leq)$. If $(X,\leq)$ is simply ordered, then the ordered set $(\mathscr{I}(X),\subseteq)$ is again simply ordered.*

(**Proof :** Assume the contrary. Then there exist $a,b\in X$ with $I(a)\not\subseteq I(b)$ and $I(b)\not\subseteq I(a)$, i. e. there exist $c\in I(a)\setminus I(b)$ and $d\in I(b)\setminus I(a)$. However, since X is simply ordered, either $c\leq d$ or $d\leq c$ which implies that either $c\in I(b)$ or $d\in I(a)$ which is impossible. •)

T4.17 (Dense Ordered Sets) An ordered set $(X,\leq)$ is called dense if for every two elements $a,b\in X$ with $a<b$, there exists an element $c\in X$ such that $a<c<b$. The set of natural numbers $(\mathbb{N},\leq)$ with its usual order is not dense, but the set $(\mathbb{Q},\leq)$ with its usual order is dense.

With this definition we have the following:

T4.18 Theorem : *Let $(X,\leq)$ be a simply ordered set. Then X is dense if and only if every non-empty initial segment of X is a lower cut of X.*

(**Proof :** ($\Rightarrow$): Let $b\in X$ be such that $I(b)\neq\emptyset$. It is enough to prove that $I(b)$ has no maximum. If $x:=\mathrm{Max}(I(b))$ exists, then $x<b$ and hence by denseness of X there exists $c\in X$ with $x<c<b$ a contradiction to the maximality of x, since $c\in I(b)$. ($\Leftarrow$): Let $a,b\in X$ with $a<b$. Then $a\in I(b)$ and hence $I(b)$ is a lower cut in X by assumption, in particular, a is not the maximum in $I(b)$, i. e. there exists $c\in I(b)$ with $a<c<b$. •)

T4.19 (Continuous Ordered Sets) An ordered set $(X,\leq)$ is called continuous if it is simply ordered, dense and conditionally complete.

With this definition we have the following:

T4.20 Theorem : *Let $(X,\leq)$ be a continuous ordered set and let $L\subseteq X$. Then L is a lower cut of X if and only if L is a proper initial segment of X.*

(**Proof :** ($\Rightarrow$): Let L be a lower cut in X. Since X is conditionally complete and $L\neq\emptyset$, $a:=\mathrm{LUB}_X(L)$ exists. Now, since X is simply ordered, either $x\leq a$ or $a\leq x$ for every $x\in X$. Therefore, since L is a lower cut, we have $a \notin L$ and $x < a$ for every $x \in L$, i. e. $L \subseteq I(a)$, moreover, $y \notin L$ if $a \leq y$. ($\Leftarrow$): Immediate from Theorem T4.18. •)

T4.21 Theorem : *Let $\mathscr{L}(X)$ be the set of lower cuts of an ordered set $(X, \leq)$. If X is simply ordered and dense, then the ordered set $(\mathscr{L}(X), \subseteq)$ is continuous.*

(**Proof :** In view of Theorem T4.12, it is enough to prove that $\mathscr{L}(X)$ is dense. For this, let $L_1 \subsetneq L_2$ be lower cuts in $\mathscr{L}(X)$. Then there is $x \in L_2 \setminus L_1$ and there is $y \in L_2$ with $x < y$, since L_2 has no maximum. But then $I(y)$ is a lower cut in X by Theorem T4.18 and $L_1 \subsetneq I(y) \subsetneq L_2$. •)

T4.22 Let $(X, \leq)$ be an ordered set.

(a) Let $\mathscr{I}(X)$ be the set of all initial segments in X. Suppose that X is simply ordered. Is the ordered set $(\mathscr{I}(X), \subseteq)$ conditionally complete?

(b) Let $\mathscr{L}(X)$ be the set of all lower cuts in X. Suppose that X is dense. Is the ordered set $(\mathscr{L}(X), \subseteq)$ dense?

(c) Give an example of an ordered set $(X, \leq)$ such that the ordered set $(\mathscr{I}(X), \subseteq)$ of initial segments of X is complete and simple ordered.

T4.23 (W e l l O r d e r e d S e t s) Ordered sets as well as simply ordered sets lack the important property, namely, the existence of the least element, or, even more general property that *every non-empty subset has the least element*. For example, the ordered set

$$(X, \leq) = \{ (\ldots ; a_3 ; a_2 ; a_1), (\ldots ; b_3 ; b_2 ; b_1 ; \ldots ; c_3 ; c_2 ; c_1) \}$$

has no least element. Also none of the simply ordered subsets (chains) $A = \{a_1, a_2, a_3, \ldots\}$, $B = \{b_1, b_2, b_3, \ldots\}$ or $C = \{c_1, c_2, c_3, \ldots\}$ has least element. We therefore the need to make the following definition:

An ordered set $(X, \leq)$ is called w e l l o r d e r e d or an o r d i n a l (or the order $\leq$ is called an w e l l o r d e r i n g on X) if every non-empty subset of X has the least element.

Clearly, every well ordered set is simply ordered, but not conversely. For example, the ordered set $(\mathbb{Z}, \leq) = \{ (\ldots ; -2 ; -1 ; 0 ; 1 ; 2 ; \ldots) \}$ is simply ordered but not well ordered. The dual of an well ordered set need not be well ordered for example, the ordered set $(\mathbb{N}, \leq) = \{ (0 ; 1 ; 2 ; \ldots) \}$ is well ordered, but its dual $(\mathbb{N}, \leq)^{\rm op} = \{ (\ldots ; 2 ; 1 ; 0) \}$ is not well ordered (it has no minimum). Even the subset of negative integers has no least element.

Let $(X, \leq)$ be a finite non-empty ordered set. Then X has (at least one) a minimal and (at least one) a maximal element. Further, prove that X is simply ordered if and only if X is well ordered.

T4.24 Examples : (1) (W e l l o r d e r i n g P r o p e r t y of $\mathbb{N}$) : The ordered set $(\mathbb{N}, \leq)$ with the usual order $\leq$ is well ordered. This ordinal is usually denoted by ω. (**Remark :** This will be proved later in Lecture Notes 5 by using the axiom of induction. In fact, it is an incarnation of the axiom of induction!)

(2) Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{w\}$ where $w \notin \mathbb{N}$. Extend the usual order $\leq$ on $\mathbb{N}$ to the order $\overline{\leq}$ on $\overline{\mathbb{N}}$ by $w \overline{\leq} w$ and $n \overline{\leq} w$ for all $n \in \mathbb{N}$. Then the ordered set $\overline{\omega} := (\overline{\mathbb{N}}, \overline{\leq}) = \{ (0 ; 1 ; 2 ; \ldots ; w) \}$ is also well ordered and it has the minimum 0 and the maximum w.

(3) Each of the following order is an well ordering on $\mathbb{Z}$:

(i) $\leq_1 = \{ (0 ; 1 ; -1 ; 2 ; -2 ; 3 ; -3 ; \ldots ; n ; -n ; \ldots) \}$.

(ii) $\leq_2 = \{ (0 ; 1 ; 3 ; 5 ; 7 ; \ldots ; 2 ; 4 ; 6 ; 8 ; \ldots ; -1 ; -2 ; -3 ; -4 ; \ldots) \}$.

(iii) $\leq_3 = \{ (0 ; 3 ; 4 ; 5 ; 6 ; \ldots ; -1 ; -2 ; -3 ; -4 ; \ldots ; 1 ; 2) \}$.

These orders are quite different from each other. For example:

(a) Every non-zero element in the order in (i) has an immediate predecessor.

(b) The elements -1 and 2 in the order in (ii) have no immediate predecessors.

(c) The elements -1 and 1 in the order in (iii) have no immediate predecessors.

(d) There are no maximal elements in the orders in (i) and (ii), but the elements 2 is a maximal element in the order in (iii), in fact it the maximum. The element 0 is the least element in all the three orders.

T4.25 Let $(W, \leq)$ be an well ordered set.

(a) Every element in W other than the last element has a unique immediate successor.

(b) Every element of W has at most one immediate predecessor.

(c) Every well ordered set is conditionally complete.

(d) The ordered set $(\mathscr{I}(W), \subseteq)$ of initial segments of W is also well ordered.

T4.26 (Order-homomorphisms) Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be ordered sets. A map $f : X \to Y$ is said to be order-preserving (or order-homomorphism) if for all $x, x' \in X$ with $x \leq x'$ implies that $f(x) \leq f(x')$.
When there is no room for confusion, we shall drop (but remember them at appropriate places) the suffixes in the notations of the orders $\leq_X$ and $\leq_Y$ on X and Y, respectively. With this an order-homomorphism is also called an increasing map. Similarly, if for all $x, x' \in X$ with $x < x'$ implies that $f(x) < f(x')$, then we say that f is strictly increasing. A map is strictly increasing if and only if it is injective order-homomorphism. Compositions of order-homomorphisms is again an order-homomorphism. A bijective map $f : X \to Y$ of ordered sets is called an order-isomorphism or similarity if both f and its inverse f^{-1} are order-homomorphisms. In general, a bijective order-homomorphism need not be an order-isomorphism. Two ordered sets X and Y are called isomorphic or similar if there is an order-isomorphism $f : X \to Y$ and in this case we write $f : X \xrightarrow{\approx} Y$. An ordered-isomorphism of an ordered set X is also called an automorphism of X. The set of all automorphisms of an ordered set X is denoted by $\mathrm{Aut}\, X = \mathrm{Aut}\,(X, \leq)$. (**Remark :** Note that $(\mathrm{Aut} X, \circ)$ is a group under composition and is called the automorphism group of the ordered set X.)

(a) $\mathrm{Aut}\, \omega = \{\mathrm{id}_{\mathbb{N}}\}$. More generally, for every ordinal $(W, \leq)$ show that $\mathrm{Aut}\, W = \{\mathrm{id}_W\}$.

(b) For each integer $n \in \mathbb{Z}$, the map $\lambda_n : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x + n$, is an automorphism of the ordered set $(\mathbb{Z}, \leq)$ with inverse λ_{-n}. Moreover, $\mathrm{Aut}\,(\mathbb{Z}, \leq) = \{\lambda_n \mid n \in \mathbb{Z}\}$. (**Remark :** In other words the map $\mathbb{Z} \to \mathrm{Aut}\,(\mathbb{Z}, \leq)$ defined by $n \mapsto \lambda_n$, is an isomorphism of groups.)

(c) Let I be any set. Describe all the automorphisms of the ordered set $\mathrm{Aut}\,(\mathfrak{P}(I), \subseteq)$. (**Hint :** Each permutation $\sigma : I \to I$ defines an automorphism $\sigma_* : (\mathfrak{P}(I), \subseteq) \to (\mathfrak{P}(I), \subseteq)$, $A \mapsto \sigma(A)$ and conversely. Therefore the map $\mathfrak{S}(I) \to \mathrm{Aut}\,(\mathfrak{P}(I), \subseteq)$ defined by $\sigma \mapsto \sigma_*$, is an isomorphism of groups.)

(d) Show that the ordinals ω and $\overline{\omega}$ (see T4.24-(2)) are not isomorphic.

(e) Show that no two of the well orders on $\mathbb{Z}$ given in T4.24-(3) are isomorphic, i. e. no two of the ordered sets $(\mathbb{Z}, \leq_1), (\mathbb{Z}, \leq_2), (\mathbb{Z}, \leq_3)$, are isomorphic.

(f) Let $(X, \leq_X)$ be any order set and $f : X \to Y$ be any bijection. Then one can define an order $\leq_Y$ on Y such that the map $f : (X, \leq_X) \to (Y, \leq_Y)$ is an order-isomorphism. (**Remark :** This can be used to constructions of well orders (or more generally, orders with give properties) on countable sets X (in terms of of the bijection $f : \mathbb{N} \to X$). The notation adopted in T4.4 is correspond to the bijective maps. For instance, the examples of orders given in T4.24-(2) correspond to the bijective maps $\mathbb{N} \to \mathbb{Z}$ and then the usual order on $\mathbb{N}$ is used to define orders $\leq_i$, $i = 1, 2, 3$ on Z. What are these bijective maps $f : \mathbb{N} \to \mathbb{Z}$ exactly?. — Though Zermelo's Theorem assures every set can be well ordered, no specific constructions for well orders on any uncountable (for example, on the set of real numbers!) set in known. Moreover, there are sets on which no specific constructions of a simple order in known. For example, on the set $\mathbb{R}^{\mathbb{R}}$ of real valued functions of one real variable. Note also that a well order guaranteed by Zermelo's Theorem is obviously not unique and is not stated to have any relation to any given structure on the set. For example, a well order on the sets $\mathbb{Q}$ and on R cannot coincide with their usual orders.)

(**Remark :** Below in T4.30, T4.32 and T4.33, we give a characterizations (up to an order-isomorphism) of the special ordered sets $(\mathbb{N}, \leq)$, $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ of the natural numbers, rational numbers and real numbers with their usual orders, respectively.)

T4.27 (Invariants under order-isomorphisms) The notions of distinguished elements — predecessor, successor, immediate predecessor, immediate successor, minimum, maximum, minimal, maximal, lower bounds, upper bounds, GLB, LUB, lower cut, initial segments, simply order, dense order, continuous order and well order, are preserved under order-isomorphisms. In such case, we usually say that these properties are invariant under order-isomorphisms. The invariants are used for characterizations of special ordered sets up to order isomorphisms, see for example, T4.30, T4.32 and T4.33. (Proofs of these assertions are routine verifications and hence we leave them to the reader. — **Remark :** These invariants are very useful for concluding the given ordered sets are non-isomorphic. For example, the ordered sets in T4.24-(2) and T4.26-(d) are non-isomorphic.)

T4.28 Let $(X, \leq)$ be a simply ordered set. Then

(a) If every initial segment $I_X(x)$ is finite, then $(X, \leq)$ is well ordered. Moreover, every element $x \in X$ other than the minimum has an immediate predecessor. (**Hint :** Assume the contrary that there is a non-empty subset $A \subseteq X$ has no minimum. Let $a \in A$. Since a is not minimum of A and since A is simply ordered, there is $b \in A$ with $b < a$. Repeating this argument conclude that the initial segment $I_X(a)$ contain an infinite subset, a contradiction. The last assertion is also proved by the similar argument.)

(b) If X is non-empty and finite, then X is well ordered and the maximum $\text{Max}\,(X)$ exists.

T4.29 Two $(X, \leq_X)$ and $(Y, \leq_Y)$ non-empty finite simply ordered sets are order-isomorphic if and only if $\#X = \#Y$, i. e. there exists a bijective map $f : X \to Y$. (**Hint :** Use induction on $\#X$.)

T4.30 Theorem (Characterization on the ordinal ω) *For a non-empty simply ordered set $(X, \leq)$, the following statements are equivalent :*

(i) *X is order isomorphic to ω.*

(ii) *For every $x \in X$, the initial segment $I_X(x)$ is finite and X contain no maximum.*

(iii) *X is infinite and for every $x \in X$, the initial segment $I_X(x)$ is finite.*

(iv) *X contain minimum, no maximum and that X contain no lower cut.* (**Remark :** A proof use *Recursion Theorem*, See Lecture Notes 5 for details.)

Deduce that :

(a) Corollary : *Let $(X, \leq)$ be a non-empty simply ordered set $(X, \leq)$ which has no maximum. If every initial segment $I_X(x)$ is finite, then X is countable.* (**Hint :** Immediate from the implication (ii) $\Rightarrow$ (i) of Theorem T4.30.)

(b) Corollary : *Two $(X, \leq_X)$ and $(Y, \leq_Y)$ non-empty simply ordered sets are order-isomorphic if and only if both satisfy the condition* (ii) *of Theorem* T4.30. (**Hint :** Immediate from the implication (ii) $\Rightarrow$ (i) of Theorem T4.30.)

T4.31 Theorem : *Let $(X, \leq)$ be a countable simply ordered set and let $(Y, \leq)$ be a non-empty dense simply ordered set with no minimum and no maximum. Then $(X, \leq)$ is isomorphic to a subset of Y.* (**Remark :** A proof use *Axiom of Choice*, See Lecture Notes 4a and Lecture Notes 5 for details.)

Deduce that :

(a) Corollary : *Every non-empty dense simply ordered set $(Y, \leq)$ with no minimum and no maximum has a subset isomorphic to $(\mathbb{Q}, \leq)$.* (**Hint :** The ordinal $(\mathbb{Q}, \leq)$ is countable and simply ordered.)

(b) Corollary : *Every non-empty simply ordered set $(X, \leq)$ is isomorphic to a subset of $(\mathbb{Q}, \leq)$.* (**Hint :** The ordinal $(\mathbb{Q}, \leq)$ is dense and simply ordered and has no minimum and no maximum.)

T4.32 (Characterization on the ordinal $(\mathbb{Q}, \leq)$) *For a non-empty simply ordered set $(X, \leq)$, the following statements are equivalent :*

(i) *$(X, \leq)$ is order isomorphic to $(\mathbb{Q}, \leq)$.*

(ii) *X is countable, has no minimum, no maximum and that $(X, \leq)$ is dense.*

(**Remark :** A proof use countability of $\mathbb{Q}$ and the *Axiom of Choice*, See Lecture Notes 4a and Lecture Notes 5 for details.)

(c) Corollary : *Two* $(X,\leq_X)$ *and* $(Y,\leq_Y)$ *non-empty simply ordered sets are order-isomorphic if and only if both satisfy the condition* (ii) *of Theorem* T4.32. (**Hint :** Immediate from the implication (ii) $\Rightarrow$ (i) of Theorem T4.32.)

T4.33 Theorem : *Let* $(X,\leq)$ *be a continuous ordered set with no minimum and no maximum and let* Y *be a dense subset of* X. *Then* $(X,\leq)$ *is order-isomorphic to the ordered set* $(\mathscr{L}(Y),\subseteq)$ *of all lower cuts of* Y. (**Hint :** We may assume that $X\neq\emptyset$. Check that the map $f:X\to\mathscr{L}(Y)$, defined by $x\mapsto I_Y(x)$ is an order-isomorphism.)

Deduce that :

(a) Corollary : (Characterization on the ordinal $(\mathbb{R},\leq)$) *For a simply ordered set* $(X,\leq)$, *the following statements are equivalent :*

(i) $(X,\leq)$ *is order isomorphic to* $(\mathbb{R},\leq)$.

(ii) X *has no minimum, no maximum,* X *have a countable dense subset and that* $(X,\leq)$ *is continuous.*

(b) Corollary : *Two* $(X,\leq_X)$ *and* $(Y,\leq_Y)$ *simply ordered sets are order-isomorphic if and only if both satisfy the condition* (ii) *of Corollary* (a) *above.* (**Hint :** Immediate from the implication (ii) $\Rightarrow$ (i) of The Corollary (a) above.)

(**Remark :** (The set of real numbers) The set of rational numbers $(\mathbb{Q},\leq)$ with its usual order is a dense simply ordered set, but its not conditionally complete (and hence not continuous). For example, the subset $\{x\in\mathbb{Q}\mid x^2\leq 2\}$ is bounded by 2 and has no least upper bound. The set $\mathbb{R}=(\mathscr{L}(\mathbb{Q}),\subseteq)$ of all the lower cuts of a dense simply ordered set $(\mathbb{Q},\leq)$ is conditionally complete dense simply ordered set and hence continuous, see Theorem T4.21. Motivated by this, a real number is, by definition, a lower cut of the set $\mathbb{Q}$ of rational numbers with its usual order.)

T4.34 (Isomorphisms of Well ordered sets) We study questions related to the isomorphisms between well ordered sets. Since every set can be well ordered (see Lecture Notes 4a); therefore the study of well ordered sets has special significance.

(1) Let $(X,\leq_X)$ be a simply ordered set. Then $(X,\leq_X)$ is well ordered if and only if X has no subset which is isomorphic to the ordinal $\omega^{\rm op}$. (**Proof :** ($\Rightarrow$): Follows easily from T4.27. ($\Leftarrow$): Let $A\subseteq X$ be non-empty and assume on contrary that A has no minimum. Then for every $a\in A$, the initial segments $A_a\neq\emptyset$ for all $a\in A$. Let $f:A\to\cup_{a\in A}A_a$ be a choice function of the family $\{A_a\mid a\in A\}$ (this exists by the *Axiom of Choice*, See Lecture Notes 4a). Then for each $a\in A$, f induces an order-isomorphism $\omega^{\rm op}\xrightarrow{\approx}\{a,f(a),f(f(a)),\ldots,\}=\{f^n(a)\mid n\in\mathbb{N}\},(n\mapsto f^n)(a))$. •)

(2) Let $(X,\leq)$ be an ordered set. A section of X is a subset A of X with the following property: if $x\in A, y\in X$ with $y\leq x$, then $y\in A$. — The subsets $\emptyset$ and X are sections of X. Arbitrary intersections and arbitrary unions of sections of X are again sections of X. For $a\in X$, the subsets $I_X(a)=X_a:=\{x\in X\mid x<a\}$ and $\overline{X}_a:=\{x\in X\mid x\leq a\}$ are sections of X.

(a) The map $(X,\leq)\to(\mathfrak{P}(X),\subseteq)$, $a\mapsto\overline{X}_a$, is an order-homomorphism and induces an order-isomorphism of X onto a subset of $\mathfrak{P}(X)$.

(b) Suppose further that $(X,\leq)$ is well-ordered. Then for every section A, $A\neq X$, of X, there exists exactly one $a\in X$ such that $A=X_a$. The map $a\mapsto X_a$, is an order-isomorphism of X onto the set of sections different from X which is ordered by the natural inclusion. The set of sections of X is well-ordered and has a greatest element. (**Hint :** Let $a:=\mathrm{Min}(X\setminus A)$. Then $A=X_a$, since A is a section.)

(3) Let $(X,\leq)$ be a well–ordered set and let $g:X\to X$ be an injective order-homomorphism. Then $x\leq g(x)$ for all $x\in X$. (**Proof :** If $A:=\{x\in X\mid g(x)<x\}\neq\emptyset$, then $a:=\mathrm{Min}(A)\in A$, i. e. $g(a)<a$ and hence $g(g(a))<g(a)<a$, i. e. $g(a)\in A$ a contradiction to the minimality of a. •)

(4) If $\mathrm{Im}\,g$ is a section of X, then $g=\mathrm{id}_X$. In particular, id_X is the only isomorphism of X onto itself. (**Proof :** If $\mathrm{Img}(g)\subseteq X$, then $\mathrm{Img}(g)=X_a$ for a unique $a\in X$. But then $g(a)<a$ a contradiction to part (a). Therefore $\mathrm{Img}(g)=X$, i. e. g is bijective. Now, since X is well ordered and hence simply ordered,

by Exercise 4.9-(c) g is an automorphism, in particular, $x \leq g^{-1}(x)$ for all $x \in X$ by part (3) and hence $g(x) \leq g(g^{-1}(x)) = x$ for all $x \in X$. altogether $g(x) = x$ for all $x \in X$, i. e. $g = \mathrm{id}_X$. See also T4.26-(a). •)

(5) For every $a \in X$, there is no injective order-homomorphism $X \to X_a$. In particular, X is not isomorphic to any of its initial segment X_a, $a \in X$. (**Proof :** Assume the contrary that there is an injective order-homomorphism $f : X \to X_a$ for some $a \in X$. Then by part (a) $x \leq f(x)$ for all $x \in X$, in particular, $f(a) \notin X_a$ a contradiction. •)

(6) Give an example of a well ordered set $(X, \leq_X)$ which is order isomorphic to a proper subset Y of X. (**Hint :** The successor map $\sigma : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $n \mapsto n+1$, is an order-isomorphism.)

(7) If $(X, \leq_X)$ and $(Y, \leq_Y)$ are two well–ordered sets, then there is at most one order-isomorphism of X onto a section of Y.

T4.35 (Principle of Transfinite Induction) One of the significant features of well ordering $\leq$ on a set X is that it enables us to extend the principle of mathematical induction for $(\mathbb{N}, \leq)$ (or the axiom of induction of $\mathbb{N}$) to arbitrary well ordered $(W, \leq)$. This feature is called the *theorem* (or *principle*) of *transfinite induction*. This provides a convenient device for making definitions or to carry out constructions at $a \in W$ depending upon what has been defined or done at all the predecessors of a in the well ordering $\leq$. The general form of this process is described in the theorem below :

Theorem (Principle of Transfinite Induction) *Let $(W, \leq)$ be a well ordered set and let $V \subseteq W$. Suppose that every element $a \in W$ satisfies the following condition: if every predecessor of a belongs to V, then $a \in V$ (in symbols: for every $a \in W$, if $I_X(a) \subseteq V$, then $a \in V$). Then $V = W$.* (**Proof :** Assume that $(W \setminus V) \neq \emptyset$ and $a := \mathrm{Min}\,(W \setminus V)$. Then by the minimality of a, we have $I_W(a) \subseteq V$ and hence $a \in V$ a contradiction. •)

(**Remark :** In the case of the ordinal ω the principle of transfinite induction is also known as second principle of mathematical induction and is equivalent to the first principle of mathematical induction. In this equivalence the most crucial property used is that each element of ω has an immediate predecessor. Since well ordered sets do not always have this property the analogue of the first induction principle is not generally true. For example, on the ordinal $\overline{\omega}$ in Example T4.24-(2), one cannot establish the first induction principle!. See more on this in the Lecture Notes 5.)