Lecture Notes

Hand written notes which are pasted below is the subsection **2.B F i n i t e S e t s**.

2.B FINITE SETS

Let $n \in \mathbb{N}$. Prototype for a finite set with n elements is the set $\{1,2,\cdots,n\} = \{x \in \mathbb{N} \mid 1 \leq x \leq n\}$ of the first n positive natural numbers.

We say that a set X is a finite set with n elements if there exists a bijective map $\{1,2,\cdots,n\} \longrightarrow X$; equivalently elements of X are enumerated by the numbers $1,2,\cdots,n$, where distinct elements are numbered by distinct numbers, i.e. $X = \{x_1,\cdots,x_n\}$ with $x_i \neq x_j$ for $i \neq j$. The number n is determined uniquely (see 2.B.3) and is called the cardinal number or cardinality of X and is denoted by $|X|$ or $\mathrm{Card}\, X$.

2.B.1 Theorem Let X be a finite set. Then every injective map $f : X \longrightarrow X$ is bijective.

Proof By assumption there is a bijective map $\{1,\cdots,n\} \xrightarrow{\varphi} X$. Since $f : X \longrightarrow X$ is injective, the composite map $\tilde{f} = \varphi^{-1} \circ f \circ \varphi$:

$$\{1,2,\cdots,n\} \xrightarrow{\varphi} X \xrightarrow{f} X \xrightarrow{\varphi^{-1}} \{1,2,\cdots,n\}$$

is also injective. Further, $\tilde{f}$ is surjective if and only if f is surjective. Therefore replacing X by $\{1,\cdots,n\}$ and f by $\tilde{f}$, we may assume that $X = \{1,\cdots,n\}$.

Now, by using induction on n, we shall prove the surjectivity of f. If $n=0$, then $[1,n] = \emptyset$ and every map from $[1,n]$ into $[1,n]$ is bijective and hence in this case nothing to prove. Suppose that $m>0$; the assertion is trivial if $n=1$, so assume $m>1$. Put $I := [1,n-1] := \{1,\cdots,n-1\}$. We consider the two cases $f(I) \subseteq I$ and $f(I) \nsubseteq I$ separately.

Case 1 : $f(I) \subseteq I$. In this case by induction hypothesis $f(I) = I$. Now, $f(n) \notin I$, since f is injective. Therefore $f(n) = n$. This proves that $\operatorname{im} f = \{1, 2, \cdots, n\}$, i.e. f is surjective, and hence f is bijective.

Case 2 : $f(I) \not\subseteq I$, i.e. there exists $a \in I$ with $f(a) = n$. Then $f(n) \neq n$, since f is injective and so $b := f(n) \in I$. Define a new map $F : \{1, \cdots, n-1, n\} \longrightarrow \{1, \cdots, n-1, n\}$ by $F(i) = f(i)$ if $i \in \{1, \cdots, n\} \setminus \{a, n\}$, $F(n) = n$ and $F(a) = b$. Then clearly F is injective and $F(I) \subseteq I$. Therefore by case 1, F is bijective. Now, the map $\tilde{f} := f F^{-1} f : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}$ is injective and $\tilde{f}(n) = (f F^{-1} f)(n) = (f F^{-1})(f(n)) = (f F^{-1})(b) = f(F^{-1}(b)) = f(a) = n$, and so $\tilde{f}$ is bijective by case 1, in particular, $\tilde{f}$ is surjective. Therefore f is surjective. This proves that f is bijective.

2.B.2 Corollary Let $m, n \in \mathbb{N}$ with $m < n$. Then every map $f : \{1, \cdots, n\} \longrightarrow \{1, \cdots, m\}$ is not injective.

Proof Let $\iota : \{1, \cdots, m\} \longrightarrow \{1, \cdots, n\}$ be the canonical inclusion. Suppose that there is an injective map $f : \{1, \cdots, n\} \longrightarrow \{1, \cdots, m\}$. Then the composite map $\iota \circ f : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}$ is also injective and hence surjective by 2.B.1. But clearly $\operatorname{im}(\iota \circ f) \subseteq \operatorname{im} \iota = \{1, \cdots, m\} \subsetneq \{1, \cdots, n\}$ a contradiction.

2.B.3 Corollary Let $m, n \in \mathbb{N}$. If there exists a bijective map $f : \{1, \cdots, m\} \longrightarrow \{1, \cdots, n\}$, then $m = n$.

Proof Suppose that $f : \{1, \cdots, m\} \longrightarrow \{1, \cdots, n\}$ is bijective. Then both f and f^{-1} are injective. Therefore $m \leq n$ and $n \leq m$ and so $m = n$.

2.B.4 Corollary (Pigeonhole principle) Let X, Y be finite sets with $|X| > |Y|$ and let $f : X \longrightarrow Y$ be any map. Then f is not injective.

Proof Let $m := |X|$ and $\varphi : \{1, \cdots, m\} \longrightarrow X$ be a bijective map. Let $n := |Y|$ and $\psi : \{1, \cdots, n\} \longrightarrow Y$ be a bijective map. Then $m > n$ and $\psi^{-1} \cdot f \cdot \varphi$ is not injective by 2.B.2. Therefore f is not injective.

2.B.5 Proposition Let $X \xrightarrow{f} Y$ be a surjective map. If X is finite then Y is also finite and $|Y| \leq |X|$.

Proof Let $m := |X|$ and $\varphi : \{1, \cdots, m\} \longrightarrow X$ be a bijective map. We prove the assertion by induction on m. If $m = 0$, then $X = \phi$ and so $Y = f(X) = \phi$. The assertion is trivial for $m = 1$ also. Now, assume that $m \geq 2$ and assume the assertion for $m-1$. Put $X' := X \setminus \{\varphi(m)\}$ and $Y' := f(X') \subseteq Y$. Then $f' := f/X' : X' \longrightarrow Y'$ is surjective and so Y' is finite and $|X'| \geq |Y'|$ by induction hypothesis. Put $k := |Y'|$ and let $\psi' : \{1, \cdots, k\} \longrightarrow Y'$ be a bijective map.

Case 1 $\quad f(\varphi(m)) \in Y'$, i.e. $Y' = Y$ is finite and $|Y| = k \leq |X'| \leq |X|$.

Case 2 $\quad f(\varphi(m)) \notin Y'$, i.e. $Y = Y' \cup \{f(\varphi(m))\}$. In this case the map $\psi : \{1, \cdots, k+1\} \longrightarrow Y$ defined by

$$\psi(i) = \begin{cases} \psi'(i), & \text{if } 1 \leq i \leq k, \\ f(\varphi(m)), & \text{if } i = k+1, \end{cases}$$

is bijective, i.e. $|Y| = k+1$ and $|X| = m = 1 + m - 1 \geq 1 + k = |Y|$.

2.B.6 Corollary Images of a finite set are finite.

2.B.7 Corollary Let $f : X \longrightarrow Y$ be an injective map. If Y is finite, then X is also finite and $|X| \leq |Y|$.

Proof We may assume that $X \neq \phi$. Since f is injective, $Y \neq \phi$ and hence there exists a map $g : Y \longrightarrow X$ such that $g \circ f = id_X$. Therefore g is surjective and so by 2.B.5 X is finite with $|X| \leqslant |Y|$.

2.B.8 Corollary Let X be a finite set. Then every subset Y of X is finite and $|Y| \leqslant |X|$. Moreover, if $Y \subseteq X$, then $Y = X$ if and only if $|Y| = |X|$.

2.B.4 Examples (1) The cardinality of the empty set is 0, i.e. $|\phi| = 0$.

(2) Let X and Y be sets and let $f : X \longrightarrow Y$ be a bijective map. Then X is finite if and only if Y is finite and in this case $|X| = |Y|$.

(3) Let X be a set. Suppose that X is decomposed into finite subsets $X_1, \cdots, X_r$ which are pairwise disjoint, i.e. $X = \biguplus_{i=1}^{r} X_i$. Then X is finite and

$$|X| = \sum_{i=1}^{r} |X_i| = |X_1| + \cdots + |X_r|$$

Proof Enough to prove (by induction) the assertion for $r=2$. Suppose that $f : \{1, \cdots, m\} \longrightarrow X_1$ and $g : \{1, \cdots, n\} \longrightarrow X_2$ are bijective maps. Then the map $h : \{1, \cdots, m+n\} \longrightarrow X_1 \uplus X_2$ defined by $h(i) = \begin{cases} f(i), & \text{if } i \leq m, \\ g(i-m), & \text{if } i > m, \end{cases}$ is also bijective.

(4) Let $X_1, \cdots, X_m$ be finite sets. Then the cartesian product $\prod_{i=1}^{m} X_i = X_1 \times \cdots \times X_m$ is also finite and

$$\left| \prod_{i=1}^{m} X_i \right| = \prod_{i=1}^{m} |X_i| = |X_1| \cdots |X_m|.$$

Proof We may assume (by induction) $m = 2$. The set $X_1 \times X_2$ is a disjoint union of the pairwise disjoint sets $\{x\} \times X_2$, $x \in X_1$, i.e. $X_1 \times X_2 = \biguplus_{x \in X_1} \{x\} \times X_2$ and $|\{x\} \times X_2| = |X_2|$ for every $x \in X_1$. Therefore by (3)

$$|X_1 \times X_2| = \sum_{x \in X_1} |X_2| = |X_1| \cdot |X_2|.$$

(5) Let X be a finite set and let $m \in \mathbb{N}$. Then

$$|X^m| = |X|^m.$$

The set X^m is the set of m-tuples $(x_1, \dots, x_m)$, $x_1, \dots, x_m \in X$, i.e. the set of all maps $\mathrm{Maps}(\{1, \dots, m\}, X)$ from $\{1, \dots, m\}$ into X.

<u>2.B.10 Theorem</u> Let I and X be finite sets with m and n elements respectively. Then the set X^I of all maps from I into X is finite with n^m elements, i.e. $|X^I| = |X|^{|I|}$.

<u>Proof</u> We may assume $I = \{1, \dots, m\}$. The assertion is proved in 2.B.9 (5).

<u>2.B.11 Corollary</u> Let I be a finite set with m elements. Then the power set $\mathfrak{P}(I)$ of I is also finite with 2^m elements, i.e. $|\mathfrak{P}(I)| = 2^{|I|}$.

<u>Proof</u> The map $\mathfrak{P}(I) \xrightarrow{e} \{0,1\}^I$, $J \longmapsto e_J =$ the indicator function of $J \subseteq I$, is bijective. Therefore by 2.B.9 (2) and 2.B.10, we have $|\mathfrak{P}(I)| = |\{0,1\}^I| = 2^{|I|}$.

<u>2.B.12 Lemma</u> (Shepherd-rule) Let $f : X \longrightarrow Y$ be a map of finite sets. Then

$$|X| = \sum_{y \in \operatorname{im} f} |f^{-1}(y)|.$$

<u>Proof</u> Note that the fibres $f^{-1}(y)$, $y \in \operatorname{im} f$ of f are pairwise disjoint and $X = \biguplus_{y \in \operatorname{im} f} f^{-1}(y)$. Therefore by 2.B.9 (3)

$$|X| = \sum_{y \in \operatorname{im} f} |f^{-1}(y)|.$$

2.B.13 Corollary Let X, Y be finite sets and let $f: X \longrightarrow Y$ be a map. Suppose that $|X| > s \cdot |Y|$ for some $s \in \mathbb{N}$. Then there exists $y \in Y$ with $|f^{-1}(y)| > s$.

Proof Suppose that $|f^{-1}(y)| \leq s$ for all $y \in Y$. Then by 2.B.11 $|X| = \sum_{y \in \operatorname{im} f} |f^{-1}(y)| \leq \sum_{y \in \operatorname{im} f} s \leq s|\operatorname{im} f| \leq s|Y| <$

$|X|$, a contradiction.

2.B.14 Corollary Let X, Y be finite sets and let $f: X \longrightarrow Y$ be a surjective map with $|f^{-1}(y)| = k$ for all $y \in Y$. Then

$$|X| = k|Y|$$

Proof Since f is surjective, $\operatorname{im} f = Y$. Now, use 2.B.12.

2.B.15 Theorem Let I and X be finite sets with m and n elements respectively. Then there are exactly

$$[n]_m := \prod_{k=0}^{m-1} (n-k) = n(n-1) \cdots (n-m+1)$$

injective maps from I into X.

Proof Let $\operatorname{Inj}(I, X) := \{f: I \longrightarrow X : f \text{ is injective}\}$. Note that if $m > n$, then $\operatorname{Inj}(I, X) = \phi$ by 2.B.2 and $[n]_m = 0$ by definition. Therefore we may assume that $m \leq n$. We shall prove that $|\operatorname{Inj}(I, X)| = [n]_m$ by induction on m. If $m = 0$, then $[n]_0 = 1$, $I = \phi$ and $\operatorname{Inj}(\phi, X) = \{\phi\}$. Therefore the induction starts. Now, assume the theorem for m and consider a set I with $m+1$ elements, a set X with n elements and assume $n \geq m+1$. Since $m+1 \geq 1$, there is an element $a \in I$. Put $I' := I \setminus \{a\}$ and consider the map

$$\Phi : \mathrm{Inj}(I, X) \longrightarrow \mathrm{Inj}(I', X), \; f \longmapsto f|I'$$

If $f' \in \mathrm{Inj}(I', X)$, then there are exactly $n-m$ elements $f \in \mathrm{Inj}(I, X)$ with $\Phi(f) = f|I' = f'$, i.e. each fibre of the map Φ has cardinality $n-m$. Further, by induction hypothesis $|\mathrm{Inj}(I', X)| = [n]_m$ and therefore by 2.B.13 we have: $|\mathrm{Inj}(I, X)| = (n-m)\,|\mathrm{Inj}(I', X)| = (n-m)\,[n]_m = [n]_{m+1}$. This completes the proof.

<u>2.B.15 Corollary</u> Let X, Y be finite sets with n elements. Then there are exactly

$$n! = [n]_n = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot \cdots \cdot n$$

bijective maps from X onto Y. In particular, there are exactly $n!$ permutations of the set X, i.e. $|\mathfrak{S}(X)| = n!$.

<u>Proof</u> Note that $\overbrace{f : X \longrightarrow Y \text{ is injective if and only if } f \text{ is bijective}}^{\text{by 2.B.1}}$. Therefore the assertion is immediate from 2.B.15.

<u>2.B.17 Example</u> (Pigeonhole principle) Let X and Y be finite sets with $|X| > |Y|$. Then there are no injective maps from X into Y, i.e. for every map $f : X \longrightarrow Y$ there are elements $x, x' \in X$, with $x \neq x'$ and $f(x) = f(x')$. This assertion is also known as (Dirichlet's) <u>pigeonhole principle</u>:

Let X, Y be finite sets of same cardinality. For a map $f : X \longrightarrow Y$ the following statements are equivalent:

(i) f is injective (ii) f is surjective (iii) f is bijective.

If f is injective, then $|Y| = |X| = |f(X)|$ and so $f(X) = Y$. Conversely, if f is not injective, then $|f(X)| < |X| = |Y|$ and hence $f(X) \subsetneq Y$, i.e. f is not surjective.

2.B.18 Remark (Combinatorial Principle) The theorem 2-B.15 is a special case of the following more general combinatorial principle:

Let X be a finite set, $m \in \mathbb{N}$ and let $A \subseteq X^m$. Suppose that $n_1, \cdots, n_m \in \mathbb{N}$ are given with the following property: if $k < m$ and $(x_1, \cdots, x_k)$ is a first part of a sequence in A, then there are exactly n_{k+1} elements $x_{k+1} \in X$ such that $(x_1, \cdots, x_k, x_{k+1})$ is a first part of a sequence in A. Then

$$|A| = n_1 \cdots n_m .$$

Proof By induction on m. For inductive step from $m-1$ to m, use $A' := \{(x_1, \cdots, x_{m-1}) \in X^{m-1} \mid \exists\, x_m \in X \text{ with } (x_1, \cdots, x_{m-1}, x_m) \in A\}$.

For $n \in \mathbb{N}$, define the number $n!$ called n-factorial recursively by $0! = 1$ and $n! = n((n-1)!)$. For natural numbers m and n with $m \leq n$, we have

$$[n]_m = \frac{n!}{(n-m)!}$$

2.B.19 Theorem Let X be a finite set with n elements. For every natural number m with $m \leq n$, we have

$$|\mathfrak{P}_m(X)| = \binom{n}{m} := \frac{[n]_m}{[m]_m} = \frac{n!}{m!\,(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{1\cdot 2 \cdot \cdots \cdot m},$$

where $\mathfrak{P}_m(X) := \{ Y \in \mathfrak{P}(X) \mid |Y| = m \}$.

Proof Let $I = \{1, \cdots, m\}$. Then every $Y \in \mathfrak{P}_m(X)$ is the image of an injective map $I \longrightarrow X$. Therefore we have a surjective map $\Phi : \mathrm{Inj}(I, X) \longrightarrow \mathfrak{P}_m(X)$. Further, for $f, g \in \mathrm{Inj}(I, X)$, $\Phi(f) = \Phi(g)$ if and only if there is

a permutation $\sigma \in \mathfrak{S}(I)$ such that $f = g\sigma$. This mean that each fibre of Φ has cardinality $|\mathfrak{S}(I)| = m! = [m]_m$. Therefore by 2.B.14 $[n]_m = |\mathrm{Inj}(I, X)| = [m]_m \cdot |\mathfrak{P}_m(X)|$, i.e.
$|\mathfrak{P}_m(X)| = \dfrac{[n]_m}{[m]_m}$.

The numbers $\binom{n}{m}$ in 2.B.19 are called Binomial-coefficients. For arbitrary (real or complex) number α and every $m \in \mathbb{N}$, one can also define:

$$[\alpha]_m := \prod_{k=0}^{m-1} (\alpha - k) = \alpha(\alpha-1)\cdots(\alpha-m+1) \quad \text{and}$$

$$\binom{\alpha}{m} := \frac{[\alpha]_m}{m!} = \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{1\cdot 2\cdot \cdots \cdot m}.$$

Further for negative integer m, we put $\binom{\alpha}{m} = 0$.
From these definitions the following rules are easy to verify:

2.B.20 For arbitrary α and all $m \in \mathbb{Z}$, we have:

(1) $\binom{\alpha}{0} = 1$. (2) $\binom{\alpha}{m+1} = \binom{\alpha}{m}\dfrac{\alpha - m}{m+1} = \binom{\alpha-1}{m}\dfrac{\alpha}{m+1}$, $m \neq -1$.

(3) $\binom{-\alpha}{m} = (-1)^m \binom{m+\alpha-1}{m}$. (4) $\binom{\alpha+1}{m} = \binom{\alpha}{m} + \binom{\alpha}{m-1}$.

2.B.21 Pascal's Triangle The formula in 2.B.20 (4) give a method for computation of the Binomial coefficients $\binom{n}{m}$, where $n, m \in \mathbb{N}$. For this one arranges these Binomial coefficients in the form of the well-known Pascal's triangle:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
 & / & & & & & & \setminus & \\
\binom{n}{0} & \cdots & \binom{n}{m-1} & & \binom{n}{m} & & \cdots & & \binom{n}{n} \\
\binom{n+1}{0} & \cdots & & & \binom{n+1}{m} & & \cdots & & \binom{n+1}{n+1}
\end{array}$$

(1) For natural numbers n and m with $m \leq n$, it follows from the definition that $\binom{n}{m} = \binom{n}{n-m}$. In fact, the complement map $\mathfrak{P}_m(\{1,\dots,n\}) \longrightarrow \mathfrak{P}_{n-m}(\{1,\dots,n\})$, $J \longmapsto J' := \{1,\dots,n\} \setminus J$ is bijective.

(2) The formula in 2.B.20 (4) for $\alpha = n \in \mathbb{N}$, can also be proved by giving a bijective map:

$$\mathfrak{P}_m(\{1,\dots,n,n+1\}) \xrightarrow{\ \Phi\ } \mathfrak{P}_m(\{1,\dots,n\}) \uplus \{J \cup \{n+1\} \mid J \in \mathfrak{P}_{m-1}(\{1,\dots,n\})\}$$

$$I \longmapsto \begin{cases} I, & \text{if } n+1 \notin I, \\ I \setminus \{n+1\}, & \text{if } n+1 \in I. \end{cases}$$

<u>2.B.22 Examples</u> (1) Let X be a finite set with n elements. For every natural number m with $0 \leq m \leq n$, there are exactly $\binom{n}{m}$ subsets of X of cardinality m, i.e. $|\mathfrak{P}_m(X)| = \binom{n}{m}$ and so $|\mathfrak{P}(X)| = \left|\biguplus_{m=0}^{n} \mathfrak{P}_m(X)\right| = \sum_{m=0}^{n} \binom{n}{m}$. Using 2.B.11, we get

$$2^n = \sum_{m=0}^{n} \binom{n}{m},$$ which can also be proved easily by using induction and 2.B.20 (4).

(2) Let m and n be natural numbers. The number of m-tuples $(x_1, \cdots, x_m) \in \mathbb{N}^m$ with $\sum_{i=1}^{m} x_i \leq n$ is equal to $\binom{n+m}{m}$.

Proof The map $(x_1, x_2, \cdots, x_m) \longmapsto (x_1+1, x_1+x_2+2, \cdots, x_1+\cdots+x_m+m)$ is bijective from the given set of m-tuples onto the set $\mathfrak{P}_m(\{1,2,\cdots,n+m\})$.

(3) The number of m-tuples $(x_1, \cdots, x_m) \in \mathbb{N}^m$ with $\sum_{i=1}^{m} x_i = n$ is equal to $\binom{n+m-1}{m-1}$. (note here is an exception $\binom{-1}{-1} = 1$)

Proof The map $(x_1, \cdots, x_{m-1}, x_m) \longmapsto (x_1, \cdots, x_{m-1})$ for $m \geq 1$, is a bijective map from the m-tuples $(x_1, \cdots, x_m)$ with $\sum_{i=1}^{m} x_i = n$ onto the set of the $(m-1)$-tuples $(x_1, \cdots, x_{m-1})$ with $\sum_{i=1}^{m-1} x_i \leq n$. Now use (2).

(4) Let $X_1, \cdots, X_r$ be pairwise disjoint finite sets with $m_1, \cdots, m_r$ elements, respectively and $X = \biguplus_{i=1}^{r} X_i$. Then the number of permutations $\sigma \in \mathfrak{S}(X)$ with $\sigma(X_i) = X_i$ for $i=1, \cdots, r$ is equal to $m_1! \cdots m_r!$.

For an r-tuple $m := (m_1, \cdots, m_r) \in \mathbb{N}^r$, put $m! = m_1! \cdots m_r!$.

(5) (Polynomial Coefficients) Let X be a finite set with n elements and let $m = (m_1, \cdots, m_r) \in \mathbb{N}^r$ with $\sum_{i=1}^{r} m_i = n$. Then the number of maps $f : X \longrightarrow \{1, \cdots, r\}$ with $|f^{-1}(i)| = m_i$, $i = 1, \cdots, r$, is $\binom{n}{m} := \frac{n!}{m!} = \frac{n!}{m_1! \cdots m_r!}$.

(6) (Binomial and Polynomial theorems) For real or complex numbers a, b and every natural number n, we have:

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m = a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + n a b^{n-1} + b^n.$$

More generally, for real or complex numbers $a_1, \dots, a_r$ and every natural number n, we have:

$$(a_1 + \dots + a_r)^n = \sum_{\substack{m=(m_1,\dots,m_r)\in\mathbb{N}^r \\ m_1+\dots+m_r=n}} \binom{n}{m} a^m ,$$

where $a^m = \prod_{i=1}^{r} a_i^{m_i} = a_1^{m_1} \cdots a_r^{m_r}$ for $m=(m_1,\dots,m_r)\in\mathbb{N}^r$

and $\binom{n}{m}$ are the polynomial coefficients as in

T6.1 (Sylvester's Sieve--formula[20]) Let $X_1,\ldots,X_n$ be finite sets. For $J\subseteq\{1,\ldots,n\}$, let $X_J:=\bigcap_{i\in J}X_i$ with $X_\emptyset:=\bigcup_{i=1}^n X_i$. Prove that

$$\sum_{J\in\mathfrak{P}(\{1,\ldots,n\})}(-1)^{|J|}|X_J|=0\,,\quad\text{i.e.}\quad |X|=\sum_{\emptyset\neq J\in\mathfrak{P}(\{1,\ldots,n\})}(-1)^{|J|-1}|X_J|\,.$$

(**Proof :** By induction on n. — **Variant :** For $k=1,\ldots,n$, let Y_k be the set of elements $x\in X_\emptyset$ which belong to exactly k of the sets $X_1,\ldots,X_n$. Then $Y_k, 1\le k\le n$ are pairwise disjoint. Using Exercise 6.2 (b) we shall show that

$$\sum_{\substack{J\in\mathfrak{P}(\{1,\ldots,n\})\\ |J|\,\text{even}}}|X_J|=\sum_{k=1}^n 2^{k-1}|Y_k|=\sum_{\substack{J\in\mathfrak{P}(\{1,\ldots,n\})\\ |J|\,\text{odd}}}|X_J|\,.$$

For $k=1,\ldots,n$, there exists $J_k:=\{j_1,\ldots,j_k\}\subseteq\{1,\ldots,n\}$ with $\#J_k=k$ and $Y_k=\{x\in X_j$ for $j\in J_k$ and $x\notin X_j$ for $j\notin J_k\}$. therefore for $J\subseteq\{1,\ldots,n\}$, $x\in X_J$ if and only if $J\subseteq J_k$ for some $k\in\{1,\ldots,n\}$. Therefore in the sum on RHS, the element x is counted exactly $\#\mathfrak{P}_{\text{even}}(J_k)=2^{k-1}$ times (exactly once for each $J\in\mathfrak{P}_{\text{even}}(J_k)$). This proves the first equality above. •)

T6.2 Let I be a finite index set with n elements and let $\sigma_i\in\mathbb{N}$ for $i\in I$, $\pi:=\prod_{i\in I}\sigma_i$, $\sigma:=\sum_{i\in I}\sigma_i$ and $\sigma_H:=\sum_{i\in H}\sigma_i$ for $H\subseteq I$. Then

$$\sum_{H\subseteq I}(-1)^{|H|}\binom{\sigma_H}{n}=(-1)^n\pi\quad\text{and}\quad\sum_{H\subseteq I}(-1)^{|H|}\binom{\sigma_H}{n+1}=\frac{(-1)^n}{2}\,(\sigma-n)\pi\,,$$

(**Proof :** We may assume $I=\{1,\ldots,n\}$. For $J\in\mathfrak{P}(I)$, let $J':=I\setminus J$. Let $X=\bigcup_{i\in I}X_i$, where X_i are pairwise disjoint subsets with $|X_i|=\sigma_i$. For a proof of the first formula consider the set $\mathfrak{P}_n(X)$ and its subsets $Y_i:=\{A\in\mathfrak{P}_n(X)\mid A\cap X_i=\emptyset\}$ and use the Sieve formula in T6.1 to find $|\bigcup_{i\in I}Y_i|$. Note $Y_\emptyset=\cup_{i\in I}Y_i$. If $J\subseteq I$, $J\neq\emptyset$, then

$$Y_J=\cap_{j\in J}Y_j=\{A\in\mathfrak{P}_n(X)\mid A\cap X_j=\emptyset\text{ for all }j\in J\}=\mathfrak{P}_n(\biguplus_{i\in J'}X_i)\quad\text{and hence}\quad \#Y_J=\binom{\sigma_{J'}}{n}.$$

Further, $\mathfrak{P}_n(X)\setminus Y_\emptyset=\mathfrak{P}_n(X)\setminus(\bigcup_{i\in I}Y_i)=\bigcap_{i\in I}(\mathfrak{P}_n(X)\setminus Y_i)=\{A\in\mathfrak{P}_n(X)\mid A\cap X_i\neq\emptyset\text{ for all }i\in I\}$ and hence

(*) $\quad\mathfrak{P}_n(X)\setminus Y_\emptyset=\{A\in\mathfrak{P}_n(X)\mid \#A\cap X_i=1\text{ for all }i\in I\}\quad$ and $\quad\#\,(\mathfrak{P}_n(X)\setminus Y_\emptyset)=\sigma_1\cdots\sigma_n=\pi\,.$

i. e. $\displaystyle\binom{\sigma}{n}=\#\mathfrak{P}_n(X)=\#Y_\emptyset+\pi\,.$

Now, by Sieve formula T6.1, we have

$$\#Y_\emptyset+\sum_{J\in\mathfrak{P}(I),J\neq\emptyset}(-1)^{\#J}\cdot\#Y_J=0\quad\text{and hence}\quad\binom{\sigma}{n}+\sum_{J'\in\mathfrak{P}(I),J'\neq I}(-1)^{n-\#J'}\cdot\binom{\sigma_J'}{n}=\pi\,.$$

Multiplying the last equality on both sides and noting $n+n-\#J'=2(n-\#J')+\#J'$, the proof of the first equality is immediate. For the proof of second equality, using similar arguments, first prove the equality :

(**) $\mathfrak{P}_{n+1}(X)\setminus Y_\emptyset=\{A\in\mathfrak{P}_{n+1}(X)\mid\#A\cap X_j=1$ for all $j\in J\subseteq I,\#J=n-1\}$ and $\#\,(\mathfrak{P}_{n+1}(X)\setminus Y_\emptyset)=\frac12(\sigma-n)\cdot\pi$. i. e. $\displaystyle\binom{\sigma}{n+1}=\#\mathfrak{P}_{n+1}(X)=\#Y_\emptyset+\frac12(\sigma-n)\cdot\pi$ and then use Sieve formula T6.1. •)

T6.3 Let $X_1,\ldots,X_n$ be finite subsets of a finite set Ω. For $\emptyset\neq J\subseteq\{1,\ldots,n\}$, let $X_J:=\bigcap_{i\in J}X_i$ and $X:=X_\emptyset:=\bigcup_{i=1}^n X_i$. Further, for $j=1,\ldots,n$, put $\xi_j:=\sum_{J\in\mathfrak{P}_j(\{1,\ldots,n\})}|X_J|$ and $\xi_0:=|\Omega|$. Prove that

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[20] This formula is attributed to Joseph Sylvester. James Joseph Sylvester (1814-1897) was an English mathematician. He made fundamental contributions to matrix theory, invariant theory, number theory, partition theory and combinatorics. He played a leadership role in American mathematics in the later half of the 19th century as a professor at the Johns Hopkins University and as founder of the American Journal of Mathematics. It is sometimes also named for Abraham de Moivre, Daniel da Silva or Henri Poincaré.

(a) $\left|\bigcap_{i=1}^{n}(\Omega\setminus X_i)\right| = \sum_{j=0}^{n}(-1)^j\xi_j$. (**Hint :** By Sieve formula T6.1, $|X| = \sum_{j=1}^{n}(-1)^{j-1}\xi_j$. Since $\cap_{i=1}^{n}(\Omega\setminus X_i) = \Omega\setminus\cup_{i=1}^{n}X_i = \Omega\setminus X$, we get $|\cap_{i=1}^{n}(\Omega\setminus X_i)| = |\Omega| - |X|$.)

(b) For $k = 1,\ldots,n$, let Y_k be the set of all those elements in X which belongs to exactly k of the subsets $X_1,\ldots,X_n$. Then show that $|Y_m| = \sum_{r=m}^{n}(-1)^{r-m}\binom{r}{m}\xi_r$ for all $1\le m\le n$. (**Hint :** Let $1\le k,m\le n$ and let m be fixed. Suppose that $x\in Y_k$ and (may) assume that $x\in X_1,\ldots,X_k$ and $x\notin X_i$ for all $k<i\le n$. If $k<m$, then $x\notin Y_m$ and hence x does not contribute anything to ξ_r for $r\ge m$. If $k=m$, then $x\in Y_m$ and in the sum on the LHS it contributes only to one term, namely, to $\binom{m}{m}\xi_m$, since $\xi_m := \sum_{J\in\mathfrak{P}_m(\{1,\ldots,n\})}|X_J|$ and only one of these intersections, namely, $X_1\cap\cdots\cap X_m$ contains x . In the remaining case $k>m$, $x\notin Y_m$ and hence x contributes nothing. On the other hand its contribution to ξ_r is $\binom{k}{r}$ (one in each $J\in\mathfrak{P}_r(\{1,\ldots,k\})$. Therefore if we let $j=r-m$, then the problem reduces to prove the identity $\sum_{j=0}^{k-m}(-1)^j\binom{m+j}{m}\binom{k}{m+j} = 0$ which is stated in Exercise 6.2-(j)-(ii).)

T6.4 The purpose of this Exercise is to give an alternative proof of the T6.3-(b). Let Ω be a finite set and let $f:\Omega\times\mathfrak{P}(\Omega)\to\mathbb{R}$ be the map defined by $(x,A)\mapsto\begin{cases}0, & \text{if } x\notin A,\\ 1, & \text{if } x\in A.\end{cases}$ (See also indicator functions Exercise Set 2, T2.26.)

Show that :

(a) For each $A\in\mathfrak{P}(\Omega)$, the map $f(-,A)$ is the indicator function e_A of A . In particular, for any two subsets $A,B\in\mathfrak{P}(\Omega)$, we have :

(1) $f(x,\Omega\setminus A) = 1 - f(x,A)$;
(2) $f(x,A\cap B) = f(x,A)\cdot f(x,B)$;
(3) $f(x,A\cup B) = f(x,A)+f(x,B)-f(x,A\cap B)$;
(4) $|A| = \sum_{x\in\Omega}f(x,A)$. (**Hint :** See the Exercise 1.??.)

(b) Let $I := \{1,2,\ldots,n\}$ and let $X_1,\ldots,X_n\in\mathfrak{P}(\Omega)$ and for each $J\in\mathfrak{P}(I)$, let $X_J := \cap_{j\in J}X_j$ (and $X_\emptyset := \Omega$). Then prove that $\sum_{J\in\mathfrak{P}_j(I)}|X_J| = \sum_{x\in\Omega}\big(\sum_{J\in\mathfrak{P}_j(I)}f(x,X_J)\big)$. (**Hint :** Use the part (a).)

(c) If an element $x\in\Omega$ belongs to exactly k of the subsets $X_1,\ldots,X_n$, then prove that

$$\sum_{J\in\mathfrak{P}_r(I)}f(x,X_J) = \binom{k}{r}.$$

(**Hint :** Here we use the understanding that $\binom{0}{0} = 1$. We may assume that $x\in X_1\cap\cdots\cap X_k$ and $x\notin X_i$ for all $k<i\le n$. For every $J\in\mathfrak{P}_r(\{1,\ldots,n\})$, $f(x,X_J) = \prod_{j\in J}f(x,X_j) = 1$ if and only if $J\subseteq\{1,\ldots,k\}$, i.e., $J\in\mathfrak{P}_r(\{1,\ldots,k\})$. This proves that LHS is equal to the cardinality $|\mathfrak{P}_r(\{1,\ldots,k\})| = \binom{k}{r}$.)

(d) For every $x\in\Omega$, show that $f(x,\cap_{i=1}^{n}(\Omega\setminus X_i)) = \sum_{j=0}^{n}(-1)^j\big(\sum_{J\in\mathfrak{P}_j(I)}f(x,X_J)\big)$. (**Hint :** For $i\in I := \{1,\ldots,n\}$, put $X_i' := \Omega\setminus X_i$. Then by (a)-(1) , (2) LHS $= \prod_{i=1}^{n}f(x,X_i') = \prod_{i=1}^{n}(1-f(x,X_i)) = 1+\sum_{j=1}^{n}(-1)^j\sum_{J\in\mathfrak{P}_j(I)}(\prod_{k\in J}f(x,X_k)) = 1+\sum_{j=1}^{n}(-1)^j\sum_{J\in\mathfrak{P}_j(I)}f(x,X_J))$. – **Remark :** Suming over the two sides of this formula as x varies over Ω and using the parts (a) and (b), we get the proof of the formula given in T6.3.)

T6.5 Prove the following (marriage) theorem :

Marriage Theorem (P. H a l l, 1935)[21] *Let Y_x , $x\in X$, be a finite family of finite sets. For every subset N of X assume that the set $Y_N := \cup_{x\in N}Y_x$ has at least $|N|$ elements, i. e., $|Y_N|\ge|N|$ for*

[21] The human interpretation which gave the folklore name (m a r r i a g e - t h e o r e m) provides the solution for the m a r r i a g e p r o b l e m which requires to match n girls with the set of n boys. For a complete match a (marriage)

every $N \in \mathfrak{P}(X)$. *Then there exists an injective map* $f : X \to Y_X$ *with* $f(x) \in Y_x$ *for every* $x \in X$.
(**Proof :** Proof by induction on $n = |X|$. The case of $n = 1$ is trivial. For the inductive step consider two cases :

Case 1: $|Y_N| > |N|$ for every subset $N \subseteq X$, $N \neq \emptyset$, $N \neq X$. In this case for a (fixed) $x \in X$, choose $y \in Y_x$ and consider the finite set $X' := X \setminus \{x\}$ and finite family $Y'_{x'} := Y_x \setminus \{y\}$, $x' \in X'$ of finite sets. Then clearly for each $N' \subseteq X'$, $Y'_{N'} = \cup_{x' \in N'} Y'_{x'}$ and $Y_{N' \cup \{x\}} = Y'_{N'} \cup \{x\}$ and hence the marriage condition $|Y'_{N'}| > |N'|$ still holds for the family $Y'_{x'} := Y_x \setminus \{y\}$, $x' \in X'$. Therefore by the inductive hypothesis, there is an injective map $f' : X' \to Y'_{X'}$ with $f'(x') \in Y'_{x'}$. Now, extend f' to the map $f : X \to Y_X$ by $f(x) = y$ and $f(x') = f'(x')$.

Case 2: There exists a subset $\emptyset \neq N \subsetneq X$, with $|Y_N| = |N|$. In this case, by the inductive hypothesis, there exists an injective (in fact bijective) map $g : N \to Y_N$. The trick is to show that $X'' := X \setminus N$ and $Y''_{x''} := Y_{x''} \setminus Y_N$, $x'' \in X''$ satisfy the marriage condition : Let $N'' \subseteq X''$ and $\tilde{N} := N'' \cup N$. Then $N \cap N'' = \emptyset$, $Y_{\tilde{N}} = Y_{N''} \cup Y_N = Y''_{N''} \uplus Y_N$ and $\#N'' + \#N = \#\tilde{N} \leq \#(Y_{\tilde{N}}) = \#(Y''_{N''} \uplus Y_N) = \#Y''_{N''} + \#Y_N = \#Y''_{N''} + \#N$ by assumptions and hence $\#N'' \leq \#Y''_{N''}$. Therefore by the inductive hypothesis, there is an injective map $X'' \to Y''_{X''}$ with $f''(x'') \in Y''_{x''}$. Now, define $f : X \to Y_X$ by $f(x) = g(x)$ for $x \in N$ and $f(x'') = f''(x'')$ for $x'' \in X''$. •)

(**Remarks :** This important theorem has many variations; these were discovered by G. Frobenius (1849-1917) a German mathematician, Denes König (1884-1944) a Hungarian mathematician, Robert Dilworth (1914-1993) an American mathematician, G. Birkhoff (1884-1944) and John von Neumann (1903-1957) an American and a Hungarian mathematicians. Below we give number of applications of marriage theorem.)

(a) Let $\mathfrak{p} = (X_1, \ldots, X_r)$ and let $\mathfrak{q} = (Y_1, \ldots, Y_r)$ be partitions of the set X into r pairwise disjoint subsets each of them with $n \geq 1$ elements. Show that $\mathfrak{p}$ and $\mathfrak{q}$ has a common representative system, i.e. there exist r distinct elements $x_1, \ldots, x_r$ in X such that each x_i belongs to exactly one of the subset $X_1, \ldots, X_r$ and exactly one of the subset $Y_1, \ldots, Y_r$.
(**Hint :** Applying the Marriage-Theorem T6.5 to the family $J_i := \{j \in \{1, \ldots, r\} \mid X_i \cap Y_j \neq \emptyset\}$, $i \in \{1, \ldots, r\}$ find a permutation $\sigma \in \mathfrak{S}_r$ such that $X_i \cap Y_{\sigma(i)} \neq \emptyset$ for every $1 \leq i \leq r$. – **Remark :** The assumption that $|X_i| = |Y_i| = n$ for all $i = 1, \ldots, r$ can be replaced by some what weaker condition : for every subset $J \subseteq \{1, \ldots, r\}$, the subset $X_J := \cup_{j \in J} X_j$ contains at most $|J|$ components $Y_1, \ldots, Y_r$ of $\mathfrak{q}$.)

(b) Let $\mathfrak{A}$ be the $n \times r$ integral matrix

$$\mathfrak{A} = \begin{pmatrix} 1 & 2 & \cdots & r \\ r+1 & r+2 & \cdots & 2r \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)r+1 & (n-1)r+2 & \cdots & nr \end{pmatrix}$$

and let $\mathfrak{B}$ be another $n \times r$ integeral with entries $1, 2, \ldots, nr$ (at arbitrary positions). Show that there exists a permutation $\sigma \in \mathfrak{S}_r$ such that for every $i = 1, \ldots, r$, the i-th column of $\mathfrak{A}$ and the $\sigma(i)$-th column of $\mathfrak{B}$ contain at least one element in common. (**Hint :** Use the part (a).)

(c) Let G be a finite group and let H be a subgroup of G. Let $G = Hy_1 \cup \cdots \cup Hy_r$ (respectively, $G = z_1 H \cup \cdots \cup z_r H$) be a right-coset (respectively, left-coset) decomposition for G. Show that there exist elements $x_1, \ldots, x_r \in G$ such that $G = Hx_1 \cup \cdots \cup Hx_r = x_1 H \cup \cdots \cup x_r H$. (**Hint :** Use the part (a).)

(d) Let X be a finite set with n elements. For $i \in \mathbb{N}$, let $\mathfrak{P}_i(X)$ be the set of all subsets Y of X with $|Y| = i$. Show that: If $i \in \mathbb{N}$ with $0 \leq i < n/2$ (respectively, with $n/2 < i \leq n$), then there exists an injective map $f_i : \mathfrak{P}_i(X) \to \mathfrak{P}_{i+1}(X)$ such that $Y \subseteq f_i(Y)$ for all $Y \in \mathfrak{P}_i(X)$ (respectively, an

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condition is necessary; the marriage condition can be formulated in several equivalent ways, for example, *For each* $r = 1, \ldots, n$ *every set of* r *girls likes at least* r *boys.* (or equivalently, *For each* $r = 1, \ldots, n$ *every set of* r *boys likes at least* r *girls.*) The marriage condition (also called Hall-condition) and the marriage theorem are due to the English mathematician Philip Hall (1904-1982). Hall was the main impetus behind the British school of group theory and the growth of group theory to be one of the major mathematical topics of the 20th Century was largely due to him. See : [P. Hall, On representatives of subsets, *J. London Math. Soc.* Vol. 10 (1935) 26- 30.]

injective map $g_i : \mathfrak{P}_i(X) \to \mathfrak{P}_{i-1}(X)$ such that $g_i(Y) \subseteq Y$ for all $Y \in \mathfrak{P}_i(X)$). (**Hint :** Let $0 \le i < n/2$. A pair $(Y,Y') \in \mathfrak{P}_i(X) \times \mathfrak{P}_{i+1}(X)$ is called *amicable* if $Y \subseteq Y'$. Let $\mathfrak{R}$ be a subset of $\mathfrak{P}_i(X)$ with $|\mathfrak{R}| =: r$. Further, let $\mathfrak{R}'$ be the set of all those $Y' \in \mathfrak{P}_{i+1}(X)$ which are amicable to at least one $Y \in \mathfrak{R}$. Put $s := |\mathfrak{R}'|$. Then $r(n-i) \le s(i+1)$ and hence $r \le s$. Now use the Marriage-Theorem T6.5.)

T6.6 (Stirling numbers and Bell's numbers) Let $n, r \in \mathbb{N}$ with $0 \le r \le n$.
(a) (Stirling numbers of second kind[22]) Let $\mathrm{S}(n,r) := |\mathfrak{Par}_r(X)|$, where $\mathfrak{Par}_r(X)$ is the set of all partitions $\mathfrak{p} = (X_1, \ldots, X_r)$ of X into r subsets. For all other pairs $(n,r) \in \mathbb{Z}^2$, we put $\mathrm{S}(n,r) = 0$.

Show that

(1) For $n \ge 1$, $\mathrm{S}(n,2) = 2^{n-1} - 1$.

(2) $\mathrm{S}(n,r) = \dfrac{1}{r!}|\mathrm{Maps}_{\mathrm{surj}}(X, \{1,\ldots,r\})| = \dfrac{1}{r!}\displaystyle\sum_{k=0}^{r}(-1)^k\binom{r}{k}(r-k)^n = \sum_{k=0}^{r}\frac{(-1)^k(r-k)^{n-1}}{k!\cdot(r-k-1)!}$.

In particular, $r! = \displaystyle\sum_{i=0}^{r}(-1)^k\binom{r}{k}(r-k)^r$.

(3) $\displaystyle\sum_{k=1}^{n} k!\cdot\binom{r}{k}\cdot \mathrm{S}(n,k) = r^n$.

(**Hint :** To prove (1) show that each fibre of the map $\mathfrak{P}(X)\setminus\{\emptyset, X\} \to \mathfrak{Par}_2(X)$ defined by $Y \mapsto (Y, X\setminus Y)$ has cardinality 2 and hence $2^n - 2 = |\mathfrak{P}(X)\setminus\{\emptyset,X\}| = 2\cdot|\mathfrak{Par}_2(X)|$ by Shepherd-rule 2.B.12. To prove (2) show that each fibre of the map $\mathrm{Maps}_{\mathrm{surj}}(X,\{1,\ldots,r\}) \to \mathfrak{Par}_r(X)$ defined by $f \mapsto (f^{-1}(1), \ldots, f^{-1}(r))$ has cardinality $r!$ and hence by the Shepherd-rule 2.B.12 and Exercise 6.4-(c), we have $r!\cdot|\mathfrak{Par}_r(X)| = |\mathrm{Maps}_{\mathrm{surj}}(X,\{1,\ldots,r\})|$. The last part follows from the equality $\pi(r,r) = 1$. For the proof of (3), compute the cardinality of each fibre of the map

$$\mathrm{Maps}(X,\{1,\ldots,r\}) \to \biguplus_{k=1}^{n} \mathfrak{P}_k(\{1,\ldots,r\}) \times \mathfrak{Par}_k(X)),\ f \mapsto (f(X), \mathfrak{p}(f)),$$

where $\mathfrak{p}(f) := (f^{-1}(i))_{i\in f(X)}$ and then use (2). – **Remarks :** The Stirling numbers appear in many other problems. Clearly $\mathrm{S}(n,r) = 0$ for $r > n$, $\mathrm{S}(n,n) = 1$, $\mathrm{S}(n,1) = 1$; $\mathrm{S}(n,n-1) = \binom{n}{2}$; a less trivial result is the formula for $\mathrm{S}(n,2)$ given in the part (1). For $r > 2$, there is no easy formula for $\mathrm{S}(n,r)$. For small values of n and r one can find $\mathrm{S}(n,r)$ by actually considering all partitions of a set with n elements. For higher values this becomes impracticable and also unreliable. The important recurrence relation given below in c) which allows us to compute a Stirling numbers by first computing the lower Stirling numbers. Consider the polynomial $F(T) := T^n - \displaystyle\sum_{k=0}^{n} k!\cdot\mathrm{S}(n,k)\cdot\binom{T}{k}$, where $\dbinom{T}{k} := \dfrac{T(T-1)\cdots(T-k+1)}{k!}$ are the binomial polynomials of degree k. Then, since $F(T)$ is a polynomial of degree $\le n$ with integer coefficients and by (3), the integers $0, 1, \ldots, n$ are $n+1$ distinct zeroes of F, we conclude that $F = 0$ and therefore the Stirling numbers of second kind are also defined by the polynomial equation $T^n = \displaystyle\sum_{k=0}^{n} k!\cdot\mathrm{S}(n,k)\cdot\binom{T}{k}$. If one takes this as the definition of the Stirlings numbers $\mathrm{S}(n,r)$ of second kind, then (1) and (3) are immediate by putting $T = 2$ and $T = r$ respectively. This also leads to the definition of the Stirling numbers of first kind : For $r, n \in \mathbb{N}$ with $0 \le r \le n$, let $\mathrm{s}(n,r) \in \mathbb{Z}$ be defined by the polynomial equation : $\dbinom{T}{n} = \dfrac{1}{n!}\cdot\displaystyle\sum_{r=0}^{n}(-1)^{n-r}\cdot\mathrm{s}(n,r)\cdot T^r$. Put $\mathrm{s}(n,r) = 0$ otherwise. For the existence of the numbers $\mathrm{s}(n,r)$ use the fact that $1, T, \ldots, T^n$ and $\binom{T}{0}, \binom{T}{1}, \ldots, \binom{T}{n}$ are two bases of the $\mathbb{Q}$-vector space $\mathbb{Q}[T]_n$ of polynomials with rational coefficients of degree $\le n$.)

(b) The Stirling numbers of second kind satisfy the recursion relations :

$$\mathrm{S}(0,r) = \delta_{0r}, \quad \text{and} \quad \mathrm{S}(n+1,r) = r\mathrm{S}(n,r) + \mathrm{S}(n,r-1),$$

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[22] James Stirling (1692-1770) was a Scottish mathematician whose most important work Methodus Differentialis in 1730 is a treatise on infinite series, summation, interpolation and quadrature.

where δ_{ij} denote the *Kronecker's delta*. (**Hint :** From $\binom{T}{k+1} = T\cdot\binom{T}{k} - k\cdot\binom{T}{k}$, we get $T^{n+1} = \sum_{k=0}^{n} k!\cdot \mathrm{S}(n,k)\cdot T\cdot\binom{T}{k} = \sum_{k=0}^{n+1} k!\cdot[k\cdot\mathrm{S}(n,k)+\mathrm{S}(n,k-1)]\cdot\binom{T}{k}$. — **Remark :** The Stirling numbers of first kind satisfy the recursion relations : $\mathrm{s}(0,r)=\delta_{0r}$, and $\mathrm{s}(n+1,r) = n\cdot\mathrm{s}(n,r)+\mathrm{s}(n,r-1)$.)

(c) (Bell's numbers[23]) Let X be a finite set with n elements. The number of equivalence relations on X is called the n-Bell number β_n, i. e., $|\mathfrak{Eq}(X)| = \beta_n$.

(i) The numbers β_n satisfy the recursion relations $\beta_0 = 1$ and $\beta_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\beta_k$ for all $n\in\mathbb{N}$.

(ii) Let $m,n\in\mathbb{N}$ with $m\le n$ and let $\beta_{m,n} := \sum_{i=0}^{m}\binom{m}{i}\beta_{n-i}$. Then $\beta_{0,n}=\beta_n$, $\beta_{0,n+1}=\beta_{n,n}$ and $\beta_{m+1,n+1} = \beta_{m,n}+\beta_{m,n+1}$ for all $m,n\in\mathbb{N}$ with $m\le n$.

(iii) Using the above formulas we have the following table :

n	0	1	2	3	4	5	6	7	8	9	10
β_n	1	1	2	5	15	52	203	877	4140	21147	115975

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(iv) Prove that $\beta_n = \sum_{r=0}^{n}\mathrm{S}(n,r)$ for every $n\in\mathbb{N}$. (**Hint :** See T6.?-(?) and use T6.?-(?).))

(d) Prove that $\mathrm{S}(n+1,r) = \sum_{k=1}^{n}\binom{n}{k}\mathrm{S}(k,r-1) = \sum_{k=0}^{n} r^{n-k}\mathrm{S}(k,r-1)$.

(**Hint:** For the first equality consider the map $\biguplus_{k=0}^{k}\big(\biguplus_{I\in\mathfrak{P}_k(X)}\{I\}\times\mathfrak{Par}_{r-1}(I)\big)\longrightarrow\mathfrak{Par}_k(X\biguplus\{y\})$ defined by $(I,(I_1,\ldots,I_{r-1}))\mapsto((X\setminus I)\uplus\{y\},I_1,\ldots,I_{r-1})$. The second equality is proved by induction and using recursion relations (see part (b)) : $\mathrm{S}(n+1,r) = r\,\mathrm{S}(n,r)+\mathrm{S}(n,r-1) = \sum_{k=0}^{n-1} r^{n-k}\,\mathrm{S}(k,r-1)+\mathrm{S}(n,r-1) = \sum_{k=0}^{n} r^{n-k}\,\mathrm{S}(k,r-1)$.)

[23]Eric Temple Bell (1883-1960) was a Scottish mathematician and attended Bedford Modern School where excellent mathematics teaching gave him his life-long interest in the subject. In particular, his interest in number theory came from this time. Bell wrote several popular books on the history of mathematics. He also made contributions to analytic number theory, Diophantine analysis and numerical functions. The American Mathematical Society awarded him the Bôcher Prize in 1924 for his memoir, Arithmetical paraphrases which had appeared in the Transactions of the American Mathematical Society in 1921. Although he wrote 250 research papers, including the one which received the Bôcher Prize, Bell is best remembered for his books, and therefore as an historian of mathematics. Bell did not confine his writing to mathematics and he also wrote sixteen science fiction novels under the name John Taine.