MA-219 Linear Algebra

3. Generating systems, Linear independence, Bases

August 25, 2003 ; Submit solutions **before 11:00AM ; September 1, 2003.**

Let K denote a field.

3.1. a). Let K be a field of characteristic $\neq 2$, i.e. $1+1 \neq 0$ K and let $a \in K$. Compute the solution set of the following system of linear equations over K.

$$\begin{aligned} ax_1 + x_2 + x_3 &= 1 \\ x_1 + ax_2 + x_3 &= 1 \\ x_1 + x_2 + ax_3 &= 1\,; \end{aligned} \qquad \begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 + ax_3 &= 3 \\ x_1 + ax_2 + 3x_3 &= 2\,; \end{aligned}$$

For which a these systems have exactly one solution ?

b). The set of m-tuples $(b_1, \ldots, b_m) \in K^m$ for which a linear system of equations $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, \ldots, m$, over a field K has a solution is a K-subspace of K^m.

c). Let K be a subfield of the field L and let $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, \ldots, m$ be a system of linear equations over K. If this system has a solution $(x_1, \ldots, x_n) \in L^n$, then it also has a solution in K^n.

3.2. a). Let $x_1, \ldots, x_n \in V$ be linearly independent (over K) in a K-vector space V and let $x := \sum_{i=1}^{n} a_i x_i \in V$ with $a_i \in K$. Show that $x_1 - x, \ldots, x_n - x$ are linearly independent over K if and only if $a_1 + \cdots + a_n \neq 1$.

b). Let $x_1, \ldots, x_n$ be a basis of the K-vector space V and let $a_{ij} \in K$, $1 \leq i \leq j \leq n$. Show that

$$y_1 = a_{11}x_1\,,\ y_2 = a_{12}x_1 + a_{22}x_2\,,\ \ldots\ ,\ y_n = a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n$$

is a basis of V if and only if $a_{11} \cdots a_{nn} \neq 0$ ist.

c). The family $\{\ln p \mid p \text{ prime number}\}$ of real numbers is linearly independent over $\mathbb{Q}$.

3.3. Let K be an infinite field and let $K[t]$ resp. $K[t]_m$, $m \in \mathbb{N}$ be the K-vector space of all polynomial functions on K. resp. of all polynomial functions of $\deg < m$.

a). For every $n \in \mathbb{N}$, let $f_n : K \to K$ be a polynomial function of degree $\leq n$ on K. Show that f_n , $n \in \mathbb{N}$, is a basis of the K-vector space $K[t]$ if and only if $\deg f_n = n$ for all $n \in \mathbb{N}$. (**Hint :** It is enough to prove that : for every $m \in \mathbb{N}$, $f_0, \ldots, f_{m-1}$ is a K-basis of the subspace $K[t]_m$ if and only if $\deg f_n = n$ for $n = 0, \ldots, m-1$.)

b). Let a_n, $n \in \mathbb{N}^*$ be a sequence of elements in K. Show that : for every $m \in \mathbb{N}$, the polynomial functions $1\,, t - a_1, \ldots, (t-a_1)\cdots(t-a_{m-1})$ form a K-basis of $K[t]_m$. Deduce that : the polynomial functions $(t-a_1)\cdots(t-a_n)$, $n \in \mathbb{N}$ form a K-basis of $K[t]$.

3.4. a). Let $f : I \to K$ be a K-valued function with $f(I)$ infinite image. Then the sequence f^n, $n \in \mathbb{N}$ of powers of f is linearly independent (over K) in the K-vector space K^I.

b). The sequences $(1, \lambda, \lambda^2, \ldots, \lambda^n, \ldots) \in K^{\mathbb{N}}$, $\lambda \in K$, are linearly independent over K.

3.5. a). The vector space of all sequences $K^{\mathbb{N}}$ has no countable generating system over K. (**Hint :** Consider the cases K countable and uncountable seperately to show that $K^{\mathbb{N}}$ is never countable and use exercises T3.2-c), d) and 3.5-b))

b). Let I be an infinite set. Then the K-vector space K^I of K-valued functions on I has no countable generating system over K.

c). The K-subspace of $K^{\mathbb{N}}$ generated by the characteristic functions e_A , $A \subseteq \mathbb{N}$ has no countable generating system. (**Hint :** *If $\mathcal{K}$ is a totally ordered subset of $\mathfrak{P}(\mathbb{N}) \setminus \{\emptyset\}$, then the family e_A , $A \in \mathcal{K}$ is linearly independent.* Now, use the fact that there are uncountable totally ordered subsets in the ordered set $\mathfrak{P}(\mathbb{N}), \subseteq)$.)

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

Test-Exercises

T3.1. Let $x_1, \ldots, x_n, x$ be elements of a vector space over a field K. Then

a). The family $x_1, \ldots, x_n$, $x_1 + \cdots + x_n$ is linearly dependent over K, but every n of these vectors are linearly independent over K.

b). Show that $x_1, \ldots, x_n, x$ are linearly independent over K if and only if $x_1, \ldots, x_n$ are linearly independent and $x \notin Kx_1 + \cdots + Kx_n$.

c). Show that $x_1, \ldots, x_n$ is a generating system of V if and only if $x_1, \ldots, x_n$, x is a generating system of V and $x \in Kx_1 + \cdots + Kx_n$.

T3.2. Let V be a vector space over a field K.

a). Suppose that V has a finite (resp. a countable) generating system. Then every generating system of V has a finite (resp. a countable) generating system.

b). Suppose that V has a countable infinite basis. Then every basis of V is countable infinite.

c). Suppose that there is an uncountable linearly independent system in V. Then no generating system of V is countable.

d). Suppose that K is counable and V has a countable generating system. Then V is countable. In particular, every $\mathbb{Q}$-basis of $\mathbb{R}$ is uncountable.

e). Let v_i , $i \in I$, be a generating system for V. Then every maximal linearly independent subsystem of v_i , $i \in I$, is a basis of V.

T3.3. Let K be a field.

a). Which of the following systems of functions are linearly independent over $\mathbb{R}$ in the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all functions.
1) $1, \sin t, \cos t$. 2) $\sin t, \cos t, \sin(\alpha + t)$ $(\alpha \in \mathbb{R}$ fixed$)$.
3) $t, |t|, \operatorname{Sign} t$. 4) $e^t, \sin t, \cos t$.

b). Let f_i , $i \in I$, and g_j , $j \in J$, be linearly independent K-valued functions on the sets X resp. Y. Then the functions $f_i \otimes g_j : (x, y) \longmapsto f_i(x)\, g_j(y)$, $(i, j) \in I \times J$, are linearly independent in $K^{X \times Y}$.

T3.4. Let $\lambda_1, \ldots, \lambda_n$ be pairwise distinct elements in a field K. Then the elements

$$x_1 := (1, \lambda_1, \lambda_1^2, \ldots, \lambda_1^{n-1}), \ldots, x_n := (1, \lambda_n, \lambda_n^2, \ldots, \lambda_n^{n-1}) \in K^n$$

are linearly independent over K.(**Hint:** Induction on n. Assume the result for $n-1$ and $a_1x_1 + \cdots + a_nx_n = 0$. Then we have the equations: $a_1\lambda_n x_1' + \cdots + a_n\lambda_n x_n' = 0$ and $a_1\lambda_1 x_1' + \cdots + a_n\lambda_n x_n' = 0$, and so $a_1(\lambda_n - \lambda_1)\, x_1' + \cdots + a_{n-1}(\lambda_n - \lambda_{n-1})\, x_{n-1}' = 0$, where $x_i' := (1, \lambda_i, \ldots, \lambda_i^{n-2})$, $i = 1, \ldots, n$.)

T3.5. **a).** Let $I \subseteq \mathbb{R}$ be an interval which contain more than one point. Then none of the $\mathbb{K}$-vector space $C_{\mathbb{K}}^{\alpha}(I)$, $\alpha \in \mathbb{N} \cup \{\infty, \omega\}$, has a countable generating system.

b). The $\mathbb{K}$-vector space of all convergent power series $\sum_{n=0}^{\infty} a_n x^n$ with coefficients a_n from $\mathbb{K}$ has no countable generating system over $\mathbb{K}$.

T3.6. Let $K \subseteq L$ be a field extension and let b_i , $i \in I$, be a K-basis of L. If V is a L-vector space with L-basis y_j , $j \in J$, then $b_i y_j$, $(i, j) \in I \times J$, is a K-basis of V.