# MA-219 Linear Algebra

## 4. Dimension of vector spaces

August 29, 2003 ; Submit solutions **before 11:00AM ; September 8, 2003.**

Let $K$ denote a field.

**4.1.** Let $V$ be a $K$-vector space of dimension $n \in \mathbb{N}$.

**a).** If $H_1, \ldots, H_r$ are hyperplanes in $V$, then $\mathrm{Dim}_K (H_1 \cap \cdots \cap H_r) \geq n - r$ .

**b).** If $U \subseteq V$ is a subspace of codimension $r$, then there exist $r$ hyperplanes $H_1, \ldots, H_r$ in $V$ such that $U = H_1 \cap \cdots \cap H_r$ .

**c).** Let $V$ be a $\mathbb{C}$-vector space of dimension $n \in \mathbb{N}^*$ and let $H$ be a real hyperplane in $V$ (i.e. a real subspace of dimension $2n - 1$). Then $H \cap \mathrm{i}H$ is a complex hyperplane in $V$, where $\mathrm{i}H := \{\mathrm{i}x \mid x \in H\}$. (i.e. a complex subspace of dimension $n - 1$).

**4.2.** Let $x_1 = (a_{11}, \ldots, a_{1n}), \ldots, x_n = (a_{n1}, \ldots, a_{nn})$ be elements of $\mathbb{K}^n$ with

$$|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ji}| \,.$$

Show that $x_1, \ldots, x_n$ is a basis of $\mathbb{K}^n$. (**Hint:** It is enough to show the linear independence of $x_1, \ldots, x_n$. For this, suppose that $b_1 x_1 + \cdots + b_n x_n = 0$ with $|b_i| \leq 1$ for all $i$ and $b_{i_0} = 1$ for some $i_0$ a contradiction.)

**4.3.** Let $x_1, \ldots, x_n \in \mathbb{Z}^n$ be arbitrary vectors with integer components. For every $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$, the vectors $x_1 + \lambda e_1, \ldots, x_n + \lambda e_n$ form a basis of $\mathbb{Q}^n$. (**Hint:** Suppose that $a_1(x_1 + \lambda e_1) + \cdots + a_n(x_n + \lambda e_n) = 0$ with $a_i \in \mathbb{Z}$, $\gcd(a_1, \ldots, a_n) = 1$ a contradiction.)

**4.4.** Let $K$ be a field with at least $n$ elements ($n \in \mathbb{N}^*$) and let $V$ be a finite dimensional $K$-vector space. Let $U_1, \ldots, U_n$ be subspaces of $V$ od equal dimension $r$ and let $u_{1i}, \ldots, u_{ir}$ be a basis of $U_i$ for $i = 1, \ldots, r$ . Show that there exists $\mathrm{Dim}_K V - r$ vectors in $V$ such that which simultaneously extend the given bases of $U_i$ to a basis of $V$. (**Hint:** Use exercise 2.2.)

**4.5.** Let $v_1, \ldots, v_n$ be a basis of the $n$-dimensional $K$-vector space $V$, $n \geq 1$, and let $H$ be a hyperplane in $V$. Show that there exist an index $i_0$, $1 \leq i_0 \leq n$, and elements $a_i \in K$, $i \neq i_0$ such that $v_i - a_i v_{i_0}$ , $i \neq i_0$ is a basis of $H$.

**4.6.** Let $\omega \in \mathbb{R}_+^\times$. For $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, let $f_{a,\varphi} : \mathbb{R} \to \mathbb{R}$ be defined by $t \mapsto a \sin(\omega t + \varphi)$ . Let $W := \{f_{a,\varphi} \mid a, \varphi \in \mathbb{R}\}$.

**a).** Find a $\mathbb{R}$-basis of the $\mathbb{R}$-subspace $W$ of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$. What is the dimension $\mathrm{Dim}_{\mathbb{R}} W$ ? (**Remark:** Elements of $W$ are called harmonic oscillations with the circular frequency )

**b).** Show that non-zero $f \in W$ has a unique representation

$$f(t) = a \sin(\omega t + \varphi)\,, \qquad a > 0,\ 0 \leq \varphi < 2\pi \,.$$

(**Remark:** This unique $a$ is called the amplitude and $\varphi$ is called the phase angle of $f$. The zero function has the amplitude 0 and an arbitrary phase angle. )

**c).** If $f, g \in W$, then compute the amplitude and the phase angle of the functions $f \pm g$.

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On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

## Test-Exercises

**T4.1.** Let $K$ be a field. For which $(a, b) \in K^2$, the vectors $(a, b)$ , $(b, a)$ form a basis of $K^2$ ?

**T4.2.** Let $K$ be a finite field with $q$ elements.

**a).** The multiples $m \cdot 1_K$, $m \in \mathbb{Z}$ of $1_K$ form a subfield $K'$ of $K$.

**b).** There exists a smallest positive nautral number $p$ with $p \cdot 1_K = 0$. This is a prime number and is called the c h a r a c t e r i s t i c of $K$. The field $K' \subseteq K$ contains exactly $p$ distinct elements $0, 1_K, \ldots, (p-1)1_K$ .

**c).** $q = p^n$, where $n := \mathrm{Dim}_{K'} K$ . (**Remark:** *The number of elements in a finite field is therefore a power of a prime number.* Conversely, for every power $q$ of a prime number there exists a field (which is essentially unique) with $q$ elements. We shall prove this assertion later. )

**d).** If $V$ is a $K$-vector space of dimension $n \in \mathbb{N}$, then $V$ has exactly $q^n$ elements.

**T4.3.** Let $x_i$ , $i \in I$, be a family of vectors in a $K$-vector space $V$ and let $U$ be the subspace of $V$ generated by $x_i$ , $i \in I$. Then $U$ is finite dimensional if and only if there exists a natural number $n \in \mathbb{N}$ such that every $n+1$ vectors from $x_i$ , $i \in I$ are linearly dependent. Moreover, if this condition is fullfilled then the dimension $\mathrm{Dim}_K U$ is the minimum of $n \in \mathbb{N}$ which satisfy the above condition.

**T4.4.** Let $V$ be a finite dimensional $K$-vector space and let $U$ be a subspace of $V$. Suppose that the basis $u_1, \ldots, u_m$ of $U$ is extended to a basis $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$ of $V$. Then

$$x = a_1 u_1 + \cdots + a_m u_m + b_{m+1} u_{m+1} + \cdots + b_n u_n \in V$$

is an element of $U$ if and only if the coordinates $b_{m+1} = u^*_{m+1}(x)$ , ..., $b_n = u^*_n(x)$ of $x$ with respect to the basis $u_1, \ldots, u_n$ of $V$ are zero. (**Remark:** This is the most easiest method to determine elements of a subspace. )

**T4.5.** Let $n \in \mathbb{N}^*$ and let $a_0, \ldots, a_n$ be real numbers with $a_0 < a_1 < \cdots < a_n$ .

**a).** Let $U$ be the $\mathbb{R}$-vector space of continuous piecewise linear [1]) real valued functions os the closed interval $[a_0 , a_n]$ in $\mathbb{R}$ with partition points $a_1, \ldots a_{n-1}$. Show that the functions $|t - a_0|, \ldots, |t - a_n|$ is a $\mathbb{R}$-basis of $U$. In particular, $\mathrm{Dim}_K U = n + 1$.

**b).** Let $V$ be the $\mathbb{R}$-vector space of the continuous piecewise linear functions $\mathbb{R} \to \mathbb{R}$ with partition points $a_0, \ldots, a_n$. Show that the functions $(a_0 - t)_+$ , $|t - a_0|, \ldots, |t - a_n|$ , $(t - a_n)_+$ is a basis of $V$, where $f_+ := \mathrm{Max}\,(f, 0)$ denote the positive part of a real valued function $f$. In particular, $\mathrm{Dim}_K V = n + 3$.

**c).** Let $W$ be the $\mathbb{R}$-vector space of the continuous piecewise linear functions $[a_0 , a_n] \to \mathbb{R}$ with partitions points $a_1, \ldots, a_{n-1}$ , and which vanish at both the end points $a_0$ and $a_n$. Show that there exist functions $f_1, \ldots, f_{n-1} \in W$ and the functions $g_1, \ldots, g_{n-1} \in W$ which form bases of $W$ such that the graphs of $f_i$ and $g_i$ are:

**d).** Let $k, m \in \mathbb{N}$ with $k < m$. The set of $k$-times continuously differentiable $\mathbb{R}$-valued functions on the closed interval $[a_0 , a_n]$ , which are polynomial functions of degree $\le m$ on every subinterval $[a_i , a_{i+1}]$, is a $\mathbb{R}$-vector space of dimension $(m-k)n + k + 1$ with basis

$$1,\ (t-a_0), \ldots, (t-a_0)^m ,$$

$$\big((t-a_1)_+\big)^{k+1}, \ldots, \big((t-a_1)_+\big)^m, \ldots, \big((t-a_{n-1})_+\big)^{k+1}, \ldots, \big((t-a_{n-1})_+\big)^m .$$

(**Remark:** The elements of this vector space are called s p l i n e f u n c t i o n s of type $(m, k)$ on $[a_0 , a_n]$ with partition points $a_1, \ldots, a_{n-1}$. )

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[1]) Let $n \in \mathbb{N}^*$ and let $a_0, \ldots, a_n$ be real numbers with $a_0 < a_1 < \cdots < a_n$ . A continuous real valued function $f : [a_0, a_n] \to \mathbb{R}$ is called p i e c e w i s e l i n e a r w i t h p a r t i t i o n p o i n t s $a_0, \ldots, a_n$ if $f\big|[a_i, a_{i+1}] \to \mathbb{R}$ is linear (see below) for every $i = 1, \ldots, n-1$.

A real valued function $f : [a, b] \to \mathbb{R}$ defined on the closed interval $[a, b] \subseteq \mathbb{R}$ is called l i n e a r if there exist $\lambda, \mu \in \mathbb{R}$ such that $f(t) = \lambda t \mu$ for every $t \in [a, b]$.