MA-219 Linear Algebra

5. Affine spaces and affine subspaces

September 05, 2003 ; Submit solutions **before 11:00AM ; September 15, 2003.**

Let K denote a field and let E be an affine space over K .

5.1. Let P, Q, R, S be points in the K-affine space.

a). The 4-tuple (P, Q, R, S) of points in E is called a (non-degenerate) quadrilateral if no three of the points P, Q, R, S are collinear. For an arbitrary quadrilateral (P, Q, R, S), the following assertions are equivalent:

(1) The lines PQ and SR are parallel and the lines PS and QR are parallel.

(2) $\overrightarrow{PQ} = \overrightarrow{SR}$. (3) $\overrightarrow{PS} = \overrightarrow{QR}$. (**Hint:** Use T5.1-h))

If any of these three equivalent assertions is true, then the quadrilateral (P, Q, R, S) is called a parallelogram. If $2 = 1_K + 1_K \neq 0$ in K, i.e. Char $K \neq 2$, then the above three assertions are further equivalent to

(4) The diagonals PR and QS bisect each other, i.e., (the unique) point of intersection is the midpoint of both (P, Q) and (R, S).

b). Suppose that $2 = 1_K + 1_K \neq 0$ in K, i.e. Char $K \neq 2$. Let (P, Q, R, S) be an arbitrary quadrilateral in E. Then the midpoints M_1, M_2, M_3, M_4 of the line-segments (P, Q), (Q, R), (R, S), (S, P) form a paralleogram and the diagonals M_1M_3 and M_2M_4 intersect in the center of mass $\frac{1}{4}P + \frac{1}{4}Q + \frac{1}{4}R + \frac{1}{4}S$ of the points P, Q, R, S with equal weights 1. (The center of mass $\frac{1}{2}L + \frac{1}{2}N$ of the points L, N with equal weights 1 is called the midpoint of the line-segment $(L, N) \subseteq E$.)

5.2. Let (P, Q) and (R, S) be the line-segments on parallel lines in a K-affine space with $P \neq Q$. Show that there exists a unique scalar $\lambda \in K$ such that $\overrightarrow{RS} = \lambda\overrightarrow{PQ}$. This scalar λ is called the ratio of the line-segment (R, S) to the line-segment (P, Q) and is denoted by $(R, S) : (P, Q)$ or $\frac{(R,S)}{(P,Q)}$.

a). Draw the graph of the function $\lambda : [P, Q) := \{t\overrightarrow{PQ} + P \mid t \in [0, 1)\} \to K$, $R \mapsto \frac{(P,R)}{(R,Q)}$.

b). Let S be the center of mass of the two distinct points P and Q with weights a and b, respectively. Suppose that $a + b \neq 0$ and $a \neq 0$. Then $\frac{(P,S)}{(S,Q)} = \frac{b}{a}$, i.e., S divide the line-segment (P, Q) in the ratio which is inversely proportional to the weights.

c). (Theorems of Thales[1]) Let (O, P, Q) and (O, P', Q') be non-degenerate triangles such that the points O, P, P' resp. O, Q, Q' are collinear. Show that
(1) if the lines PQ and $P'Q'$ are parallel, then $\frac{(O,P')}{(O,P)} = \frac{(O,Q')}{(O,Q)} = \frac{(P',Q')}{(P,Q)}$ and $\frac{(P,P')}{(O,P)} = \frac{(Q,Q')}{(O,Q)}$.
(2) (converse of (1)): If either $\frac{(O,P')}{(O,P)} = \frac{(O,Q')}{(O,Q)}$ or $\frac{(P,P')}{(O,P)} = \frac{(Q,Q')}{(O,Q)}$, then the lines PQ and $P'Q'$ are parallel.

5.3. Let $(P_0, \ldots, P_n)$ be a non-degenerate n-simplex, $n \geq 2$, in the real affine space E. Suppose that every point P_i is given with a positive weight a_i, $i = 0, \ldots, n$. Let S_i denote the center of mass of the i-th face (the $(n-1)$- simplex obtained from the n-simplex $(P_0, \ldots, P_n)$ by removing the point P_i) of the simplex $(P_0, \ldots, P_n)$.

a). The line-segments $S_i P_i$, $i = 0, \ldots, n$, intersect in the center of mass S of the points $P_0, \ldots, P_n$ with the weights $a_0, \ldots, a_n$.

b). For $i = 0, \ldots, n$, the ratio $\frac{(S_i, S)}{(S, P_i)} = a_i / b_i$, where b_i is the sum of the weights a_j, $j \neq i$. (**Remark:** If all the weights are equal, then this ratio is $1/n$ and in this case the point S is called the barycenter of the $P_0, \ldots, P_n$.)

5.4. Let $F_1 = U_1 + P_1$ and $F_2 = U_2 + P_2$ be two affine subspaces of a K-affine space E.

a). Show that $F_1 \cap F_2 \neq \emptyset$ if and only if $\overrightarrow{P_1 P_2} \in U_1 + U_2$.

b). The affine hull of $F_1 \cup F_2$ is called the join-space of F_1 and F_2 and is denoted by $F_1 \vee F_2$. Show that $F_1 \vee F_2 = (U_1 + U_2 + K \cdot \overrightarrow{P_1 P_2}) + P_1$.

c). (Dimension Formula) If F_1 and F_2 are finite dimensional and if $F_1 \cap F_2 \neq \emptyset$, then

$$\operatorname{Dim}(F_1 \vee F_2) + \operatorname{Dim}(F_1 \cap F_2) = \operatorname{Dim} F_1 + \operatorname{Dim} F_2 .$$

(How does one modify this formula in the case $F_1 \cap F_2 = \emptyset$?)

d). If K has more than two elements and if $F_1 \cap F_2 \neq \emptyset$, then $F_1 \vee F_2$ is the union of all line-segments $P_1 P_2$ with $P_1 \neq P_2$ and $P_1 \in F_1$, $P_2 \in F_2$. (**Remark:** But in the case $F_1 \cap F_2 = \emptyset$ an analogous assertion is not true in general.)

On the next pages one can see (simple) test-exercises; their solutions need not be submitted.

[1]) THALES: Greek geometry appears to have started in an essential way with the work of THALES of Miletus in the first half of the sixth century B. C.. This versatile genius, declared to be one of the "seven wise men" of antiquity, was a worthy founder of systematic geometry and is the first known individual with whom the use of deductive methods in geometry is associated. Thales sojourned for a times in Egypt and brought back geometry with him to Greece, where he began to apply to the subject the deductive procedures of Greek philosophy. He is credited with a number of very elementary geometrical results, the value of which is not to be measured by their content but rather by the belief that he supported them with a certain amount of logical reasoning instead of intuition and experiment. For the first time a student of geometry was committed to a form of deductive resoning, partial and incomplete though it may have been. Moreover, the fact that the first deductive thinking was done in the field of geometry, instead of algebra for instance, inaugurated a tradation in mathematics which was maintained until very recent times!

Test-Exercises

Let K be a field and let V be a K-vector space and let E be an affine space over V. Usually V and E have nothing in common. A lettered arrow x connecting two points P and Q in E simply records *pictorially* [2]) that the vector x of V sends the point P to the point Q.

T5.1. Let $x, y \in V$ and let $P, Q, R, S \in E$. Then

a). $Q = \overrightarrow{PQ} + Q$.

b). For $x \in V$, let $\tau_x : E \to E$ denote the translation $P \mapsto x + P$ of E. Let $\mathrm{T}(E)$ be the set of all translations of E. Then the map $V \to \mathrm{T}(E)$ defined by $x \mapsto \tau_x$ is bijective.

c). $x + P = Q$ if and only if $P = -x + Q$. In particular, $-\overrightarrow{PQ} = \overrightarrow{QP}$.

d). If $x + P = P$ then $x = 0$.

e). $\overrightarrow{P(x+P)} = x$, $\overrightarrow{P(x+Q)} = x + \overrightarrow{PQ}$ and $\overrightarrow{(x+P)Q} = \overrightarrow{PQ} - x$

f). $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ and $\overrightarrow{PQ} + R = \overrightarrow{PR} + Q = (\overrightarrow{PQ} + \overrightarrow{PR}) + P$.

g). $\overrightarrow{(x+P)(x+Q)} = \overrightarrow{PQ}$ and $\overrightarrow{(x+P)(y+Q)} = \overrightarrow{(x+P)Q} + y = (\overrightarrow{PQ} - x + y$.

h). $\overrightarrow{PQ} = \overrightarrow{SR}$ if and only if $\overrightarrow{PS} = \overrightarrow{QR}$.

T5.2. (Affine subspaces) A subset F of E is called a (K-)affine subspace of E if either $F = \emptyset$ or $F = U + P := \{x + P \mid x \in U\}$, where $P \in E$ and U is a K-subspace of V.

a). In the representation $F = U + P$ of a non-empty affine subspace $F \subseteq E$, the point P can be chosen any point of F and U is precisely the set of vectors $x \in V$ with $x + F = \tau_x(F) = F$, i.e., the isotropy group of F. In particular, U is uniquely determined by F.

b). Let $F_1 = U_1 + P_1$ and $F_2 = U_2 + P_2$ be non-empty affine subspaces of E, where P_1, P_2 are points in E and U_1, U_2 are subspaces of V. Then $F_1 = F_2$ if and only if $U_1 = U_2$ and $\overrightarrow{P_1P_2} \in U_1$ $(= U_2)$.

c). Every non-empty affine subspace $F = U + P$ of E is itself an affine space over the K-vector space $\mathrm{V}(F) = U$. The operation of V on E induces (by restriction) to an operation of U on F. In particular, every non-empty affine subspace has the dimension. The dimension of empty affine subspace is defined to be -1. If E is finite dimensional, then the difference $\mathrm{Dim}\, E - \mathrm{Dim}\, F$ is called the codimension of F in E and is denoted by $\mathrm{Codim}_E F = \mathrm{Codim}\,(F, E)$. The empty affine subspace has codimension $1 + \mathrm{Dim} E$. Further, $\mathrm{Dim}\, F + \mathrm{Codim}\,(F, E) = \mathrm{Dim}\, E$. Affine subspaces of the codimension 1 are called (affine) hyperplanes.

d). Let V be a K-vector space and consider it as an affine space over itself. Then the *affine* subspaces of V are (other than empty set) cosets of the form $U + y = \{x + y \mid x \in U\}$, where U is a subspace of V and y is point V. The subspaces of the vector space V are precisely the affine subspaces of V which pass through 0.

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[2]) It is very important that appropriate pictures be drawn while studying geometry. Pictures indicate what, probably, the correct theorem is and how one might attempt to prove it. Of course, the actual proof must stand on its own logical legs and be independent of figures. In geometry, in fact, in all of mathematics one should be aware of the danger of figures and then liv dangerously. Draw a lot of pictures!

The solution space of the linear system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \cdots \quad \cdots\cdots \cdots &\quad \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is an affine subspace in K^n which is parallel to the solution space L_0 of the corresponding homogeneous system of equations. Conversely, every affine subspace in K^n is the solution space of a system of linear equations over K in n unknowns.

e). Let E be an affine space over the K-vector space V, where K is a field with more than two elements. Then $F \subseteq E$ is an affine subspace of E if and only if for every pair of two distinct points of F, the line-segment joining them is also contained in F. (**Hint:** Let $F \neq \emptyset$ and $P \in F$. it is enough to show that $\{\overrightarrow{PQ} \mid Q \in F\}$ is a K-subspace of V. – **Remark:** If K has only two elements, then the affine subspaces of E are precisely those distinguished subsets F of E such that for any three distinct points P, Q, R, the affine plane PQR generated by the points P, Q, R (which contain the only point $P+Q+R$ other than P, Q, R) is contained in F.)

f). The intersection of an arbitrary family F_i, $i \in I$, of affine subspaces of E is again an affine subspace. If this intersection is non-empty and $F_i = U_i + P$, then $\bigcap_{i\in I} F_i = \left(\bigcap_{i\in I} U_i\right) + P$.

T5.3. (Affine generating systems, affinely independent and affine hulls) For every family P_i, $i \in I$, of points in E, there exists (see T5.2-f)) a smallest affine subspace F of E containing all the points P_i, $i \in I$ and is called the *affine hull of the points* P_i, $i \in I$ or the *affine subspace generated by the points* P_i, $i \in I$. In this case the family P_i, $i \in I$, is called an *(affine) generating system of* F. In fact $F = \left(\sum_{i\in I,\, i\neq i'} K \cdot \overrightarrow{P_{i'}P_i}\right) + P_{i'}$, where $i' \in I$. In particular, $1 + \mathrm{Dim}_K F \leq |I|$

Let P_i, $i \in I$ be a famliy of points in E.

a). The affine subspace F generated by the family P_i, $i \in I$ is equal to E if and only if $V = \sum_{i\in I,\, i\neq i'} K \cdot \overrightarrow{P_{i'}P_i}$, $i' \in I$; equivalently the vectors $\overrightarrow{P_{i'}P_i}$ $i \in I$, $i \neq i'$ is a generating system for the vector space V. In this case we say that the family of points $P_i, i \in I$, is an *affine generating system of* E.

b). The vectors $\overrightarrow{P_{i'}P_i}$ $i \in I$, $i \neq i'$ are linearly independent over K if and only if for every proper subset $J \subsetneq I$, the affine subspace F generated by the points P_j, $j \in I$, is a proper subset of E, i.e. $F \subsetneq E$. In this case we say that the family of points $P_i, i \in I$, is *affinely independent* over K.

c). An affinely independent generating system of points P_i, $i \in I$ of an affine space E is called an *affine basis of* E. A family of points P_i, $i \in I$ in E is an affine basis of E if and only if $I \neq \emptyset$ and for some $i' \in I$ (and hence for every $i' \in I$), the vectors $\overrightarrow{P_{i'}P_i}$ $i \in I$, $i \neq i'$ is a basis of the vector space V; equivalently $P_{i'}; \overrightarrow{P_{i'}P_i}$, $i \in I$, $i \neq i'$, is an affine coordiante system of E.

d). Let $m \in \mathbb{N}$. Then m points in E generate the affine subspace of dimension $\leq m-1$ and exactly of dimension $m-1$ if and only if they are affinely independent. Two distinct points P and Q generate the (affine) line called the *line joining* these points and is denoted by PQ. Similarly the (affine) plane generated by three affinely independent points P, Q, R is denoted by PQR.

e). Let P_i, $i \in I$, be a famliy of points in an affine space E. Show that the following are equivalent:
(i) P_i, $i \in I$, is an affine basis of E.
(ii) P_i, $i \in I$, is a maximal affine linearly independent subset in E.
(iii) P_i, $i \in I$, is a minimal affine generating system of E.

f). Let E be a finite dimensional affine space. Then every affine generating system of E contains an affine basis and every affine linearly independent system in E can be extended to an affine basis of E.

g). Let E be an affine space and let P_i, $i \in I$, be a family of points in E. Then the affine subspace of E generated by P_i, $i \in I$, is equal to the intersection of all affine hyperplanes in E, which contain these points. In particular, every affine subspace F of E is equal to the intersection of all hyperplanes in E which contain F. (**Hint:** Remember that empty union is the whole space E.)

h). (Simplexes) For $n \in \mathbb{N}$, an affinely independent $(n+1)$-tuple $(P_0, \ldots, P_n)$ of points is called an *n-simplex* in E with *vertices* $P_0, \ldots, P_n$.

1) An affine basis of an n-dimensional affine space E is an n-smplex, $n \in \mathbb{N}$ in E; every $n+1$ points which are not contained in any proper affine subspace of E form an affine basis of E. The natural number n is also called the dimension of such a simplex. One dimensional simplex is called a (line) segment, two dimensional simplex is called a triangle and three dimensional simplex is called a tetrahedron.
2) A r-dimensional affine subspace contain (non-degenerate) n-simplex if and only if $0 \le n \le r$.
3) Every simplex obtained from a given simplex by removing a vertex is called a side or face of this given simplex.
4) In K^{n+1} the n-simplex $(e_0, \ldots, e_n)$ with the vertices $e_0, \ldots, e_n$, where $e_0 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ are the elements of the standard basis of K^{n+1}, is called the standard-n-simplex over K.

T5.4. (Barycentric cocordinates) Let $P_i,\ i \in I$, be an affine basis of the affine space E and let $i' \in I$ be a fixed index. Then every point $P \in E$ has an coordinate tuple $a_i,\ i \in I,\ i \neq i'$, with respect the affine coordinate system $P_{i'};\ \overrightarrow{P_{i'}P_i}, i \neq i'$, which is determined uniquely by the equation $\overrightarrow{P_{i'}P} = \sum_{i\in I,\, i\neq i'} a_i \overrightarrow{P_{i'}P_i}$. These coordinates depend on the choice of the point $P_{i'}$ in the given affine basis $P_i,\ i \in I$; this unsymmetry is overcome by A. Möbius by introducing the barycentric coordinates. There by modifying the coordinate tuple $a_i,\ i \in I \setminus \{i'\}$, of $P \in E$ by introducing $a_{i'} := 1 - \sum_{i\in I,\, i\neq i'} a_i$ and hence extending to the I–tuple $(a_i) \in K^{(I)}$ with $\sum_{i\in I} a_i = 1$. This I–tuple is called the barycentric coordinates of P with respect to the affine basis $P_i,\ i \in I$. In the case $|I| = 3$ this is also called triangle coordinates, in the case $|I| = 4$, tetrahedron coordinates.

a). *The barycentric coordinates are independent of the choice of the point $P_{i'}$ as origin.* (**Proof** For, if $(b_i)_{i\in I}$ is the I–tuple for P, that is obtained analogously by choosing the point $P_{i''}$ as the origin. Then

$$\overrightarrow{P_{i'}P} = \overrightarrow{P_{i'}P_{i''}} + \overrightarrow{P_{i''}P} = \Big(\sum_{i\in I} b_i\Big)\overrightarrow{P_{i'}P_{i''}} + \sum_{i\in I} b_i \overrightarrow{P_{i''}P_i} = \sum_{i\in I} b_i(\overrightarrow{P_{i'}P_{i''}} + \overrightarrow{P_{i''}P_i}) = \sum_{i\in I} b_i \overrightarrow{P_{i'}P_i}\,,$$

and hence $b_i = a_i$ for $i \neq i'$ and since $\sum_i a_i = \sum_i b_i = 1$, it follows that $b_{i'} = a_{i'}$. •)

If $a_i,\ i \in I$, are the barycentric coordinates of P with respect to the affine basis $P_i,\ i \in I$, then we write $P = \sum_{i\in I} a_i P_i$.

b). The map $(a_i) \mapsto \sum_{i\in I} a_i P_i$ is a bijective map from the affine hyperplane $\{(a_i) : \sum_i a_i = 1\} \subseteq K^{(I)}$ onto E. The vectors e_i of the standard basis of $K^{(I)}$ are mapped onto the points P_i of the affine basis.

c). For an I–tuple $(a_i) \in K^{(I)}$ with $\sum_i a_i = 1$ and an *arbitrary* I–tuple $Q_i,\ i \in I$, of points in E, the point

$$\sum_{i\in I} a_i Q_i := \sum_{i\in I} a_i \overrightarrow{PQ_i} + P$$

is independent on the choice of the point $P \in E$ and hence well-defined. The affine subspace of E generated by the family $Q_i,\ i \in I$, is then simply the set of all points $\sum_i a_i Q_i,\ (a_i) \in K^{(I)},\ \sum_i a_i = 1$.

T5.5. Let P_i, $i \in I$, be a family of points in the K-affine space E.

a). Let $(a_{ij}) \in K^{(I)}$, $j \in J$, and $(b_j) \in K^{(J)}$ be tuples with $\sum_{i\in I} a_{ij} = 1$, $j \in J$; $\sum_{j\in J} b_j = 1$. Then $\sum_{j\in J} b_j(\sum_{i\in I} a_{ij}P_i) = \sum_{i\in I}(\sum_{j\in J} b_j a_{ij})P_i$.

b). For an I-tuple $(a_i) \in K^{(I)}$ with $a := \sum a_i \neq 0$, the point $S := \sum_{i\in I} \frac{a_i}{a} P_i$ is called the center of mass of the P_i, $i \in I$, with the weights a_i, $i \in I$. This point belong to the affine hull of the P_i, $i \in I$, and in the case $K = \mathbb{R}$, $a_i \ge 0$ for all $i \in I$ it also belong to the convex hull.

c). The center of mass of the P_i is the same point if the weights are multiplied by the same factor $t \neq 0$.

d). If P_i, $i \in I$, is an affine basis and $P \in E$ is a point with the barycentric coordinates a_i, $i \in I$, then P is the center of mass of the P_i, $i \in I$, with the weights a_i, $i \in I$.

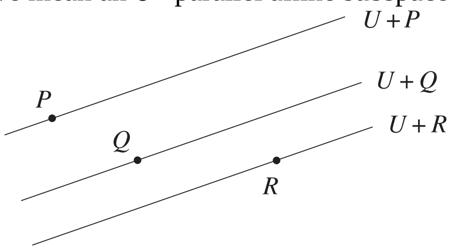
e). Let I_j, $j \in J$, be a decomposition of I with $b_j := \sum_{i\in I_j} a_i \neq 0$ for all $j \in J$ and let S_j be the center of mass of the P_i, $i \in I_j$, with weights a_i, $i \in I_j$. Then the center of mass S of the P_i, $i \in I$, with weights a_i, $i \in I$, is equal to the center of mass of the S_j, $j \in J$, with weights b_j, $j \in J$.

T5.6. (Parallelity) Two affine lines $g_1 = U_1 + P_1$, $g_2 = U_2 + P_2$ in affine plane are disjoint if and only if $g_1 \neq g_2$, but $U_1 = U_2$. In this case we say that g_1 and g_2 are parallel. In general we define the concept of parallelity of affine subspaces as follows: Two affine subspaces $F_1 = U_1 + P_1$ and $F_2 = U_2 + P_2$

with points P_1 , $P_2 \in E$ and subspaces U_1 , $U_2 \subseteq V$ are called parallel if either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$ and in this case we write $F_1 \| F_2$. The empty affine subspace is parallel to every affine subspace of E. Two disjoint affine subspaces which are not parallel are called skew.

a). Let F_1 and F_2 be two affine subspaces in E. If $F_1 \subseteq F_2$ then F_1 and F_2 are parallel. If $F_1 \| F_2$ and $F_1 \cap F_2 \neq \emptyset$, then either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.

b). If $U \subseteq V$ is a subspace, then the affine subspaces $U + P$, $P \in E$, are called U-parallel (affine) subspaces. They form (by T5.2-b)) a partition of E. If $F = U + P$ is an affine subspace of E, then by a parallel to F in E, we mean an U- parallel affine subspace.



c). Let $m \in \mathbb{N}$. On the set of all m-dimensional affine subspaces of E the parallelity is an equivalence relation. (**Remark:** The equivalence classes of this equivalence relation are called the m-dimensional directions in E. If $m = 1$, then in general the directions are not used and in the case $K = \mathbb{R}$, then the concept of the directions is very restricted: We say that two non-degenerate line-segments (P, Q) , (R, S) in E define the same direction if the lines PQ and RS are parallel and the ratio $(R, S) : (P, Q)$ is positive.)

d). Two hyperplanes in an n-dimensional affine space, $n \geq 2$, are either parallel or intersect in an affine subspace of codimension 2.

e). Let F be a finite dimensional affine subspace in an affine space E and let $P \in E$ be a point. Then there exists a unique affine subspace of dimension $\operatorname{Dim}_K F$ passing through P and which is parallel to F.

f). Let E be a finite dimensional K-affine space and let F_1 , F_2 be two disjoint non-empty affine subspaces in E. There there exist disjoint parallel affine hyperplanes H_1 , H_2 in E such that $F_1 \subseteq H_1$, $F_2 \subseteq H_2$.

T5.7. (Convex subsets, Convex hull) In the case of real affine space we have another important concept of the convexity. For two points P and Q of a real affine space E, let $[P, Q] := \{a\,P + b\,Q \mid a, b \in \mathbb{R}_+,\ a + b = 1\} = \big\{t\,\overrightarrow{PQ} + P \mid t \in [0\,,1]\big\} = \big\{t\,\overrightarrow{QP} + Q \mid t \in [0\,,1]\big\}$ be the set of points in E which lie in between P and Q.

The set of these points is called the line-segment joining P and Q, or shortly the line-segment from P to Q (or from Q to P). With this we define

A subset of a real affine space E is called convex if it contain the line-segment joining any two of its points.

Clearly affine subspaces are convex. Further arbitrary intersection of convex subsets is again convex. Therefore for an arbitrary family of points P_i , $i \in I$, in E there exists a smallest convex subset of E, which contain all these points. This convex is called the convex hull of the family P_i , $i \in I$.

a). Let P_i , $i \in I$, be a family of points in a real affine space E. The convex hull of the family P_i , $i \in I$, contain precisely the points of the form $\sum_{i\in I} a_i P_i$ with $(a_i) \in \mathbb{R}_+^{(I)}$, $\sum_{i\in I} a_i = 1$. (**Proof** The given set of points is convex, since $a\Big(\sum_{i\in I} a_i P_i\Big) + b\Big(\sum_{i\in I} b_i P_i\Big) = \sum_{i\in I}(a\,a_i + b\,b_i)\,P_i$ if $\sum a_i = \sum b_i = 1$ and $a + b = 1$, see T5.5-a), and contain all the P_i , $i \in I$. Conversely, we have to show that all the points of the given form are contained in the convex hull H. For this we may assume that I is non-empty and finite, say $I = \{0, \ldots, n\}$. We now use the induction on n. The case $n = 0$ is trivial. For the proof of $n+1$ from n, let $P = a_0 P_0 + \cdots + a_n P_n + a_{n+1} P_{n+1}$ with $a_i \geq 0$, $\sum_{i=0}^{n+1} a_i = 1$. If $a_{n+1} = 1$, then $P = P_{n+1} \in H$. Therefore assume that $a_{n+1} \neq 1$. Then by induction hypothesis $P' := \frac{a_0}{1-a_{n+1}} P_0 + \cdots + \frac{a_n}{1-a_{n+1}} P_n$ in H and hence $P = (1 - a_{n+1}) P' + a_{n+1} P_{n+1}$.

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b). For an n-simplex $(P_0, \ldots, P_n)$, the convex hull is the set of all points $a_0 P_0 + \cdots + a_n P_n$ with $a_j \geq 0$, $a_0 + \cdots + a_n = 1$. If this simplex is non-degenerate, then this set is also called the *n-simplex with vertices* $P_0, \ldots, P_n$. The convex hull of finitely many points in a real affine space E is also called a *polytope* in E.

c). The convex hull of the standard n-simplex $(e_0, \ldots, e_n)$ in K^{n+1} is the set

$$\Delta_n = \left\{ (a_0, \ldots, a_n) \in K^{n+1} : \sum_{i=0}^{n} a_i = 1, \ \ 0 \leq a_i \ \text{ for } \ i = 0, \ldots, n \right\}.$$

d). (*Theorem of Carathéodory*) Let E be an n-dimensional real affine space and let P_i, $i \in I$, be a famliy of points in E. Then the convex hull of the P_i, $i \in I$, is equal to the union of the convex hulls of all (eventually also non-degenerate) n-simplexes $(P_{i_0}, \ldots, P_{i_n})$, $i_0, \ldots, i_n \in I$. (**Hint:** Start with the case $n = 2$.)

T5.8. (*Pappus' [3]) little theorem*) Let E be an K-affine space and let g, g' be two distinct parallel lines in E and let P, Q, R (respectively P', Q', R') be points on g (respectively g') (see the figure below).

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[3]) PAPPUS : The last of the Greek geometers, lived toward the end of the third century A. D., 500 years after Appollonius and vainly strove with enthusiasm to rekindle fresh life into linguishing Greek geometry, but it proved to be the requiem of Greek geometry, for after Pappus Greek mathematics ceased to be a living study and we find merely its memory perpetuated by minor writers and commentators. EUCLID[4]), ARCHIMEDES[5]) and APPOLLONIUS[6]) mark the apogee of ancient Greek geometry and it is hardly an exaggeration to say that almost every significant subsequent geometrical development, right upto and including present times, finds its origin in some work of those three great scholars. It is therefore incumbent upon us to say at least word or two about these scholars and mathematical legacy left to them.

[4]) EUCLID : Very little is known about the life of Euclid except that he was the first professor of mathematics at the famed University of Alexandria and the father of the illustrious and long-lived Alexandrian School of Mathematics. Even dates and birthplace are not known but it seems probable that he received his mathematical training in the Platonic school at Athens.

[5]) ARCHIMEDES : One of the very greatest mathematicians of all time and certainly the greatest of antiquity, was Archimedes, a native of the Greek city of Syracuse on the island of Sicily. He was born about 287 B. C. and died during the Roman pillage of Syracuse in 212 B.C.. There is a report that he spent time in Egypt, in all likelihood at the University of Alexandria, for he numbered among his friends CONON, DOSITHEUS and ERATOSTHENES; the first two were successors of Euclid and the last was a librarian at the university. Many of Archimedes's mathematical discoveries were communicated in letters to these men.

If $PQ' \parallel QP'$ and $QR' \parallel RQ'$ then $PR' \parallel RP'$.

T5.9. Let (P, Q, P', Q') be a quadrilateral in the affine space E. Suppose that the lines PP', QQ' intersect in the point R and the lines PQ', QP' intersect in the point S (see the figure below). .

Show that $R \neq S$ and the lines PQ, $P'Q'$ are parallel if and only if the line RS and the line PQ intersect at the midpoint of (P, Q). Moreover in this case the lines RS, $P'Q'$ also intersect at the midpoint of (P', Q').

T5.10. The following constructions are given by J. Steiner and can be carried out by the straight-edge alone, i.e. any two given points are allowed to join by a line.

a). Let P, M, Q be three distinct points on a line such that the point M is the midpoint of the line-segment (P, Q). For an arbitrary point R outside the line PQ, construct a line parallel to the line PQ passing through the point R. (**Hint:** Use T5.3.)

b). Let ℓ and ℓ', $\ell \neq \ell'$ be two given parallel lines and let $P, Q \in \ell$. Then construct the middle point of the line-segment (P, Q). (**Hint:** Use T5.3.)

c). Given (in the plane of construction) a paralleogram , a line ℓ and a point P. Construct a line parallel to the line ℓ passing through P.

d). Given two points P, Q and a line ℓ parallel to the line PQ, $\ell \neq PQ$.

1) Construct the points R_1 and R_2 on the line PQ such that $\frac{(P:R_1)}{(P:Q)} = 2$ and $\frac{(P:R_2)}{(P:Q)} = 3$.

2) Construct the point R on the line PQ such that $\frac{(P:R)}{(P:Q)} = \frac{1}{3}$.

T5.11. (Theorem of Pappus) Suppose that $\dim(E) \geq 2$. Let g and g' be two distinct lines intersecting at the point S. Let P, Q, R (respectively P', Q', R') be distinct points on g (respectively on g') different from S.

If $PQ' \parallel QP'$ and $QR' || RQ'$ then $PR' || RP'$ (see the figure above).

T5.12. (Theorem of Desargues[7])) Let $P, Q, R; P', Q', R'$ be six distinct points in an affine space E with the following properties:

i). (P, Q, R) and (P', Q', R') are non-degenerate triangles.

[6] APPOLLONIUS: The third mathematical gaint of Greek antiquity who was born about 262 B.C. in Perga in southern Asia Minor. As a young man went to Alexandria and studied under the successors of EUCLID and died sometime around 200 B. C.

[7]) GIRARD DESARGUES (1591-1661) was a french mathematician.

ii). The lines PP', QQ' and RR' are distinct.

iii). $PQ||P'Q'$ and $PR \parallel P'R'$ (see the figure below).

Then $QR \parallel Q'R'$ if and only if the lines PP', QQ', RR' are parallel or intersect at a point S.

T5.13. (Theorem of Menelaus[8])) Let (A, B, C) be a triangle in an affine space E and let P, Q, R be distinct points lying on the sides BC, AC, AB respectively. Show that P, Q, R are collinear if and only if

$$\frac{(P,B)}{(P,C)} \cdot \frac{(Q,C)}{(Q,A)} \cdot \frac{(R,A)}{(R,B)} = 1\,.$$

(**Hint:** Draw a parallel line to the line BC passing through A (see the figure below).

Use the exercise 5.2-c).)

T5.14. (Theorem of Ceva[9])) Let (A, B, C) be a triangle in an affine space E and let P, Q, R be distinct points lying on the sides BC, AC, AB respectively. Show that the lines AP, BQ, CR are parallel or intersect at a point if and only if

$$\frac{(P,B)}{(P,C)} \cdot \frac{(Q,C)}{(Q,A)} \cdot \frac{(R,A)}{(R,B)} = -1\,.$$

(**Hint:** Suppose the lines intersect at a point O.

Apply the theorem of Menelaus to the triangle (A, B, P) and the line CR (respectively (A, P, C) and the line BQ (see the figure above).)

[8]) MENELAUS of Alexandria was a Greek astronomer who live in the first century A. D.

[9]) GIOVANNI CEVA (1647-1736) was an italian geometer. Theorems of Menelaus and Ceva, in their modern dress, are powerful theorems and they deal elegantly with many problems involving collinearity of poiints and currency of lines.