MA-219 Linear Algebra

6. Linear Maps

September 12, 2003 ; Submit solutions **before 11:00AM ; September 22, 2003.**

Let K be a field.

6.1. Let V and W be finite dimensional K-vector spaces. Show that

a). There is an injective K-homomorphism from V into W if and only if $\mathrm{Dim}_K V \leq \mathrm{Dim}_K W$. Deduce that a homogeneous linear system of m equations in n unknowns over K with $n > m$ has a non-trivial solution.

b). There is a surjective K-homomorphism from V onto W if and only if $\mathrm{Dim}_K V \geq \mathrm{Dim}_K W$. Deduce that a linear system $\sum_{j=1}^n a_{ij}x_j = b_i$, $i = 1, \ldots, m$ of m equations in n unknowns over K with $n < m$ has no solution for some $(b_1, \ldots, b_m) \in K^m$.

c). A homogeneous linear system $\sum_{j=1}^n a_{ij}x_j = 0$, $i = 1, \ldots, n$ of n equations in n unknowns over K has a non-trivial solution if and only if at least one of the corresponding inhomogeneous system of linear equations $\sum_{j=1}^n a_{ij}x_j = b_i$, $i = 1, \ldots, n$ has no solution.

6.2. a). Let V be a K-vector space with $\mathrm{Dim}_K V \geq 2$ (i.e. V contain at least two linearly independent vectors). Then every additive map $f : V \to V$ with $f(Kx) \subseteq Kx$ for all $x \in V$ is a homothecy of V, i.e. a multiplication ϑ_a by a scalar $a \in K$.

b). Let V be a finite dimensional K-vector space and let U, W be subspaces of V of equal dimension. Then there exists a K-automorphism f of V such that $f(U) = W$.

c). Let $f_1 : V \to V_1$ and $f_2 : V \to V_2$ be homomorphisms of K-vector spaces. The K-linear map $f : V \to V_1 \times V_2$ defined by $f(x) = \big(f_1(x)\,,\, f_2(x)\big)$ is an isomorphism if and only if f_1 surjective and $f_2|\mathrm{Ker}\, f_1 : \mathrm{Ker}\, f_1 \to V_2$ is bijective.

6.3. Let V be a finite dimensional K-vector space and let $f : V \to V$ be an endomorphism of V. Show that the following statements are equivalent:

(i) f is *not* an automorphism of V.
(ii) There exists a K-endomorphism $g \neq 0$ of V such that $g \circ f = 0$.
(ii′) There exists an K-endomorphism $g' \neq \mathrm{id}_V$ of V such that $g' \circ f = f$.
(iii) There exists an K-endomorphism $h \neq 0$ of V such that $f \circ h = 0$.
(iii′) There exists an K-endomorphism $h' \neq \mathrm{id}_V$ of V such that $f \circ h' = f$.

6.4. a). Let V be a K-vector space of countable infinite dimension. Then V and the direct sum $V \oplus V$ are isomorphic. (**Remark:** This is true for arbitrary infinite dimensional vector spaces V.)

b). Give an example of an endomorphism of a vector space (necessarily infinite dimensional) which is injective, but not surjective (resp. surjective, but not injective).

c). Let V be a K-vector space with basis x_i , $i \in I$ and let $f : V \to K$ be a linear form $\neq 0$ on V with $f(x_i) = a_i \in K$, $i \in I$. Find a basis of $\mathrm{Ker}\, f$.

6.5. Let $f_1, \ldots, f_n$ be linearly independent K-valued functions on the set D. Further, let $t_1, \ldots, t_n$ be pairwise distinct points in D and let V be the subspace of K^D (n-dimensional) generated by $f_1, \ldots, f_n$. Show that for every choice of $b_1, \ldots, b_n \in K$ the *interpolation problem*

$$f(t_1) = b_1, \ldots, f(t_n) = b_n$$

has a solution $f \in V$ if and only if the trivial problem

$$f(t_1) = \cdots = f(t_n) = 0\,.$$

has only trivial (the zero function) solution in V.

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

Test-Exercises

T6.1. Let $V := \mathbb{K}[t]$ be the $\mathbb{K}$-vector space of $\mathbb{K}$-valued polynomial functions on $\mathbb{K}$. Which of the following maps $f : V \to V$ are $\mathbb{K}$-linear? Find the bases for Ker f and im f for those f which are $\mathbb{K}$-linear.

a). $f(x) := x^{(n)} =$ (the n-th derivative of x, $n \in \mathbb{N}$.)

b). $f(x) := x(0) + \ddot{x}$.

c). $f(x) := (t \mapsto \int_0^t \tau\, \dot{x}(\tau) d\tau\,)$.

d). $f(x) := P(D)x$, where $P(t) \in \mathbb{K}[t]$ is a monic polynomial[1]) and D is the differential operator $x \mapsto \dot{x}$.

T6.2. Let $h : D \to D'$ be an arbitrary map. For every field K, the map $h^* : K^{D'} \to K^D$ defined by $g \mapsto g \circ h$ is K-linear. Describe the functions in Ker h^* and in im h^*. Show that h^* is injective (resp. surjective) if and only if h is surjective (resp. injective).

T6.3. **a).** A map $f : V \to W$ of $\mathbb{Q}$-vector spaces V and W is $\mathbb{Q}$-linear if and only if it is additive.

b). For every K–vector space V, the map $f \mapsto f(1)$ is a K-isomorphism of $\mathrm{Hom}_K(K, V)$ onto V.

c). Let K' be a subfield of the field K, V be a K'-vector space and W be a K-vector space, then W is a K'-vector space in a natural way. With this $\mathrm{Hom}_{K'}(V, W)$ is a K-subspace of W^V.

T6.4. **a).** Let f and g be endomorphisms of the finite dimensional vector space V. If $g \circ f$ is an automorphism of V, then both g and f are also automorphisms of V.

b). Let $f : V \to W$ be a homomorphism of finite dimensional K-vector spaces.

(1) Show that f injective if and only if there exists a homomorphism $g : W \to V$ such that $g \circ f = \mathrm{id}_V$.

(2) Show that f surjective if and only if there exists a homomorphism $h : W \to V$ such that $f \circ h = \mathrm{id}_W$.

T6.5. (Pointer representation) Let $\omega \in \mathbb{R}_+^\times$ and V be the $\mathbb{R}$-vector space of the functions $a \sin(\omega t + \varphi)$, $a, \varphi \in \mathbb{R}$, with basis $\sin \omega t$, $\cos \omega t$, (see exercise 4.6). Then the map

$$\gamma : a \sin(\omega t + \varphi) \longmapsto a e^{\mathrm{i}\varphi},\ a \geq 0\,,$$

is a $\mathbb{R}$-vector space isomorphism of V onto $\mathbb{C}$. (**Remark:** This isomorphism is called the pointer representation of the simple harmonic motion with the circular frequency ω. The differentiation in V correspond to the multiplication by $\mathrm{i}\omega$ to the pointer representation, i.e. $\gamma(\dot{x}) = \mathrm{i}\omega\gamma(x)$ for $x \in V$. In the representation $a e^{\mathrm{i}\varphi}$ of $a \sin(\omega t + \varphi)$, $a \geq 0$, $a = |a e^{\mathrm{i}\varphi}|$ is called the (maximal) amplitude and $e^{\mathrm{i}\varphi}$ is called the phase factor.)

T6.6. Let $I \subseteq \mathbb{R}$ be an interval with more than one point and $a \in I$.

a). For $n \in \mathbb{N}^*$, Let $T_{a,n} : \mathrm{C}_{\mathbb{K}}^{n-1}(I) \to \mathbb{K}[t]_n$ be the map which maps every function $f \in \mathrm{C}_{\mathbb{K}}^{n-1}(I)$ to its Taylor polynomial of degree $< n$ of f at a, i.e.

$$f \mapsto T_{a,n}(f) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}\,(t-a)^k\,.$$

Show that $T_{a,n}$ is $\mathbb{K}$-linear. Describe the kernel and image of $T_{a,n}$.

[1]) A polynomial $P(t) = \sum_{i=0}^n a_i t^i \in K[t]$ of degree n over a field K is called a monic polynomial if the leading co-efficient $a_n = 1$.

b). Let $T_a : \mathrm{C}^\infty_{\mathbb{K}}(I) \to \mathbb{K}[\![t-a]\!]$ be the map which maps every function $f \in \mathrm{C}^\infty_{\mathbb{K}}(I)$ to its Taylor's series of f at a, i.e.

$$T_a(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\,(t-a)^k\,.$$

Show that T_a is a $\mathbb{K}$-linear map of $\mathrm{C}^\infty_{\mathbb{K}}(I)$ in the space $\mathbb{K}[\![t-a]\!]$ of all power series in $(t-a)$ with coefficients in $\mathbb{K}$. The kernel of T_a is the space of all functions which are *plate* [2]) at a. Further, show that T_a is surjective.

(**Remark** This is precisely the following classical theorem of real analysis which is proved in 1895 by the French mathematician Borel, Émile Félix Édouard-Justin (1871-1956) in his thesis.

Theorem (Borel) *For every sequence a_n, $n \in \mathbb{N}$, of real or complex numbers there exists an infinitely many times differentiable function f on $\mathbb{R}$ with values in $\mathbb{R}$ resp. $\mathbb{C}$ such that for all $n \in \mathbb{N}$ gilt: $f^{(n)}(0) = a_n$.*

A differentiable function on interval $I \subseteq \mathbb{R}$ can be given by using its derivative f; if f is continuous, then the function ($a \in I$ be a fixed point) $\int_a^x f(t)\,dt$, upto an additive constant, is the required function. This can be generalised, for instance to give a construction of *hat-functions* which are further useful for many constructions in analysis. A function $h : \mathbb{R} \to \mathbb{R}$ is called a hat-function if it satisfies properties stated in the following theorem:

Theorem *Let $a, a', b', b \in \mathbb{R}$ with $a < a' < b' < b$. Then there exists an infinitely many times differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) = 0$ for $t \notin [a, b]$, $h(t) = 1$ for $t \in [a', b']$ and $0 < h(t) < 1$ otherwise.*

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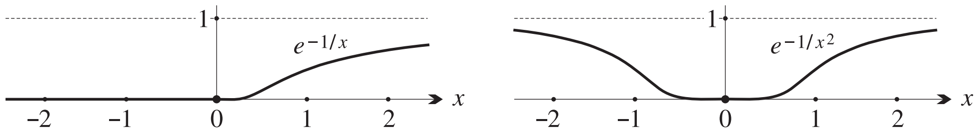
[2]) (Plate Functions) Let $f : D \to \mathbb{C}$ be an *analytic* [3]) function on an interval $D \subseteq \mathbb{R}$ or a domain $D \subseteq \mathbb{C}$. If the derivatives $f^{(n)}(a)$ of f at a point $a \in D$ are zero, then by the *Taylor's formula* [4]) the function f vanishes in a neighbourhood of a and hence by the *identity theorem*[5]) f is identically 0 on the whole D. The analogous result does *not* hold for functions defined on an interval $I \subseteq \mathbb{R}$, which are infinitely many times differentiable. An infinitely many times differentiable function $f : I \to \mathbb{C}$ is called plate at point $a \in I$, if $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. There are functions which are plate at a point, but are not indentically zero in any neighbourhood of this point. Such a function cannot be analytic; for example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-1/x}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

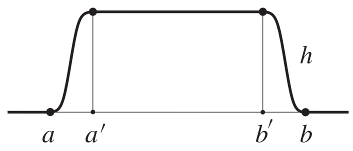
This function is infinitely many times differentiable and it is plate at 0. It is enough to show that $f|\mathbb{R}_+$ is plate at 0. For $x > 0$, we have (can be seen easily by induction on n) $f^{(n)}(x) = h_n(1/x)\exp(-1/x)$ with a monic polynomial function h_n of degree $2n$. Since $\lim_{x\to 0+} h(1/x)\exp(-1/x) = 0$ for *every* polynomial function h, the assertion follows.



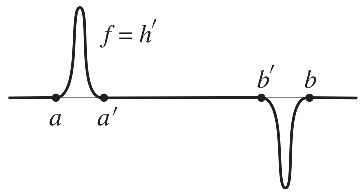
[3]) (Analytic functions) Let either D be an interval in $\mathbb{R}$ with more than one point or an open subset in in$\mathbb{C}$. A function $f : D \to \mathbb{K}$ is called analytic at a point $a \in D$, if there exists a neighbourhood U of a and a convergent power series $\sum a_k(x-a)^k$ such that $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ for all $x \in U \cap D$. – A function $f : D \to \mathbb{K}$ is called analytic in D, if f is analytic at every point of D.

[4]) Let $f = \sum_{n=0}^{\infty} a_n(x-a)^n$ be the power series expansion of the analytic function $f : D \to \mathbb{C}$ at a point $a \in D$. Then for every $m \in \mathbb{N}$ $f^{(m)} = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\, a_n(x-a)^{n-m}$ is the power series expansion of the m-th derivative of f at the point $a \in D$. All these power series have the same radius of convergence. In particular, $a_m = \frac{f^{(m)}(a)}{m!}$ for all $m \in \mathbb{N}$ (this is also known as the Taylor-formula for analytic functions.).

[5]) (Identity theorem for analytic functions) Let D be either an interval in $\mathbb{R}$ or a domain in $\mathbb{C}$. Two analytic functions on D are equal on the whole D if and only if they are equal on a seubset of D, which has at least one limit point in D.



Proof The graph of the derivative $f := h'$ of the required function is the following:



Further, we must have $\int_a^{a'} f(t)\,dt = -\int_{b'}^{b} f(t)\,dt = 1$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(t) = 0$ for $t \le 0$ and $g(t) = e^{-1/t}$ for $t > 0$. Then g is infinitely many times differentiable function. Now, let $f(t) := \big(g(t-a)g(a'-t)/c\big) - \big(g(t-b')g(b-t)/d\big)$, where $c := \int_a^{a'} g(t-a)\,g(a'-t)\,dt$ and $d := \int_{b'}^{b} g(t-b')\,g(b-t)\,dt$. Then f is the required function and the function $h(x) := \int_a^x f(t)\,dt$ has the properties stated in the assertion. •

Now using hat-functions, we can give a proof of the Borel's theorem :

Proof of Borel's theorem : Let $h : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable hat- function with $h(t) = 1$ for $|t| \le 1$ and $h(t) = 0$ for all $|t| \ge 2$, further, let $h_n(t) := t^n h(t)$, $n \in \mathbb{N}$. Then $|h_n^{(\nu)}(t)| \le M_n$ for all $t \in \mathbb{R}$ and all $\nu \in \mathbb{N}$ with $0 \le \nu \le n$. Put $b_n := |a_n| M_n + 1$ and $f_n(t) := a_n h_n(b_n t)/n! b_n^n$. Then the function $f(t) := \sum_{n=0}^{\infty} f_n(t)$ is a required function. Since $|f_n^{(\nu)}(t)| \le 1/n!$ for all $n > \nu$ and all $t \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} f_n^{(\nu)}(t)$ of ν-th derivatives is uniformly convergent [6]) for every $\nu \in \mathbb{N}$. Therefore by [7]) $f^{(\nu)}(t) = \sum_{n=0}^{\infty} f_n^{(\nu)}(t)$ and in particular, $f^{(\nu)}(0) = \sum_{n=0}^{\infty} f_n^{(\nu)}(0) = a_\nu$ for all $\nu \in \mathbb{N}$. •)

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[6]) **Uniform convergence** Let D be an arbitrary set and let (f_n) be a sequence of functions $f_n : D \to \mathbb{K}$ on D with values in $\mathbb{K}$.

(1) The sequence (f_n) is called (pointwise) convergent (on D), if there exists a function $f : D \to \mathbb{K}$ with $\lim f_n(x) = f(x)$ for all $x \in D$, i.e. if for every $x \in D$ and for every $\varepsilon > 0$ there exists (dependent on x and ε) $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| \le \varepsilon$ for all $n \ge n_0$.

(2) The sequence (f_n) is called uniformly convergent (on D), if there exists a function $f : D \to \mathbb{K}$ such that for every $\varepsilon > 0$ there exists (depending only on ε and not on x) $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| \le \varepsilon$ for all $n \ge n_0$.

Uniform convergence of the function sequence (f_n) implies its point-wise convergence. The function f with $f(x) = \lim f_n(x)$ is called the limit function or the limit of the sequence (f_n) and is denoted by $f = \lim_{n\to\infty} f_n = \lim f_n$.

For a sequence (f_n) of functions $f_n : D \to \mathbb{K}$, the sequence of partial sums $\sum_{n=0}^{k} f_n$, $k \in \mathbb{N}$, is called the series of the f_n, $n \in \mathbb{N}$. Its limit function (if it exists) it is denoted by $\sum_{n=0}^{\infty} f_n$. If the convergence of partial sums is uniform on D, then we say that the series converges uniformly on D.

[7]) **Theorem** *Let D be a domain in $\mathbb{C}$ or an interval in $\mathbb{R}$ and let $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of differentiable functions. Further, let $x_0 \in D$ be a fixed point. Suppose that:*

(1) *The sequence $f_n(x_0)$, $n \in \mathbb{N}$, is convergent.*

(2) *The sequence f_n', $n \in \mathbb{N}$, of derivatives is locally uniformly convergent* [8]) *on D.*

Then the sequence f_n, $n \in \mathbb{N}$, is locally uniformly convergent on D to a differentiable limit function $f : D \to \mathbb{C}$, and $f' = \lim_{n\to\infty} f_n'$.

[8]) A sequence $f_n : D \to \mathbb{K}$, $n \in \mathbb{N}$, of functions on $D \subseteq \mathbb{C}$ is called locally uniform convergent, if for every point $a \in D$ there exists a neighbourhood U of a such that the sequence $f_n | U \cap D$, $n \in \mathbb{N}$, is uniformly convergent on $U \cap D$.