MA-219 Linear Algebra

7. Rank Theorem

September 19, 2003 ; Submit solutions **before 11:00AM ; September 29, 2003.**

Let K be a field and let U, V, W be vector spaces over K. .

7.1. a). (Inequality of Sylvester) Let $f:U\to V$ and let $g:V\to W$ be linear maps. If U and V are finite dimensional, then $\text{Rank}\, f+\text{Rank}\, g-\text{Dim}\, V \leq \text{Rank}\,(gf) \leq \text{Min}\,(\text{Rank}\, f,\ \text{Rank}\, g)$.
(**Hint:** $\text{Rank}\,(gf) = \text{Rank}\, f - \text{Dim}\,(\text{im}\, f\cap \text{Ker}\, g)$.)

b). (Inequality of Frobenius) Let $f:U\to V,\ g:V\to W$ and let $h:W\to X$ be K-linear maps. If U, V and W are finite dimensional, then $\text{Rank}\,(hg)+\text{Rank}\,(gf)\leq \text{Rank}\, g+\text{Rank}\,(hgf)$.
(**Hint:** We may assume that g is surjective and apply the part a).)

7.2. Let $f:V\to W$ be a homomorphism of K-vector spaces. Show that $\text{Ker}\, f$ is finite dimensional if and only if there exists a homomorphism of K-vector space $g:W\to V$ and an operator $h:V\to V$ on V such that $\text{Rank}\, h$ is finite and $gf = h+\text{id}_V$.

7.3. Let $f_1,\ldots,f_r\in \text{Hom}_K(V,W)$ be K-vector space homomorphisms of finite rank. For arbitrary $a_1,\ldots,a_r\in K$, the rank of $a_1f_1+\cdots+a_rf_r$ is finite and

$$\text{Rank}\,(a_1f_1+\cdots+a_rf_r)\leq \text{Rank}\, f_1+\cdots+\text{Rank}\, f_r\ .$$

7.4. Let V and W be finite dimensional K-vector space and let V', W' be subspaces of V resp. W. Show that there exists a K-homomorphism $f:V\to W$ with $\text{Ker}\, f = V'$ and $\text{im}\, f = W'$ if and only if $\text{Dim}\, V'+\text{Dim}\, W' = \text{Dim}\, V$.

7.5. a). Let $a_{ij}\in K$, $i=1,\ldots,m,\ j=1,\cdots,n$. Then the linear system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \cdots\cdots\cdots\cdots & \\ a_{m1}x_1+\cdots+a_{mn}x_n &= b_m \end{aligned}$$

over K has a solution in K^n for every $(b_1,\ldots,b_m)\in K^m$ if and only if its rank is m. Moreover, in this case the solution space is an affine subspace of dimension $n-m$.

b). Let $s,n\in\mathbb{N},\ s\leq n$. Then every affine subspace of K^n of dimension s is a solution space of a linear system of equations of rank $n-s$ in n unknowns and $n-s$ equations.

c). Let $f_i : V_i\to V_{i+1},\ i=1,\cdots,r$, be surjective K-vector space homomorphism with finite dimensional kernels. Then the composition $f := f_r\circ\cdots\circ f_1$ from V_1 to V_{r+1} also has finite dimensional kernel and $\text{Dim}_K\text{Ker}\, f = \sum_{i=1}^r \text{Dim}_K\text{Ker}\, f_i$. (**Remark:** For example: Let $P(x) = (x-\lambda_1)\cdots(x-\lambda_n)$ be a polynomial in $\mathbb{C}[x]$ with (not necessarily distinct) zeros $\lambda_1,\ldots,\lambda_n\in\mathbb{C}$. Then the differential operator $P(D) = (D-\lambda_1)\cdots(D-\lambda_n)$ on $\mathrm{C}^\infty_{\mathbb{C}}(I)$, where $I\subseteq\mathbb{R}$ is an interval has n-dimensional kernel, since $D-\lambda$ is surjective for every $\lambda\in\mathbb{C}$ with a finite dimensional kernel $\mathbb{C}e^{\lambda t}$, .)

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

Test-Exercises

T7.1. Let f and g be endomorphisms of the finite dimensional K-vector space V with $g \circ f = 0$. Then $\operatorname{Rank} f + \operatorname{Rank} g \leq \operatorname{Dim} V$. In particular, if $f^2 \, (= f \circ f) = 0$, then $\operatorname{Rank} f \leq \frac{1}{2} \operatorname{Dim} V$.

T7.2. Let $g : V \to W$ be K-linear and let V' be a subspace of V. If V is finite dimensional, then $\operatorname{Dim} V - \operatorname{Dim} V' \geq \operatorname{Rank} g - \operatorname{Rank} (g|V')$.

T7.3. Let f be an operator on the finite dimensional K-vector space V of odd dimension. Then $\operatorname{im} f \neq \operatorname{Ker} f$.

T7.4. Let f be an operator on the finite dimensional K-vector space V.

a). The following statements are equivalent :

(1) $\operatorname{Ker} f = \operatorname{im} f$. (2) $f^2 = 0$ and $\operatorname{Dim} V = 2 \cdot \operatorname{Rank} f$.

b). The following statements are equivalent:

(1) $\operatorname{Rank} f = \operatorname{Rank} f^2$. (1′) $\operatorname{im} f = \operatorname{im} f^2$. (2) $\operatorname{Dim} \operatorname{Ker} f = \operatorname{Dim} \operatorname{Ker} f^2$. (2′) $\operatorname{Ker} f = \operatorname{Ker} f^2$.
(3) $\operatorname{im} f \cap \operatorname{Ker} f = 0$. (4) $\operatorname{im} f + \operatorname{Ker} f = V$.
(**Remark :** (3) and (4) together mean that V is the direct sum of $\operatorname{im} f$ and $\operatorname{Ker} f$.)

T7.5. Let $f : U \to V$ and let $g : V \to W$ be homomorphisms of K-vector spaces. If one of these homomorphism have a finite rank, then the composition $g \circ f$ also has a finite rank. If f is surjective (resp. g is injective), then $\operatorname{Rank} (g \circ f) = \operatorname{Rank} g$ (resp. $\operatorname{Rank} (g \circ f) = \operatorname{Rank} f$).

T7.6. Let $f : V \to W$ be a homomorphism of K-vector spaces and let u_i, $i \in I$, be a basis of $\operatorname{Ker} f$. Then for a family v_j, $j \in J$, of vectors of V, the family $f(v_j)$, $j \in J$, of the image vectors is a basis of $\operatorname{im} f$ if and only if the families u_i, $i \in I$; v_j, $j \in J$, together form a basis of V.