MA-219 Linear Algebra

8. Direct sums and projections

September 24, 2003 ; Submit solutions **before 11:00 AM ; October 03, 2003.**

Let K be a field and let U, V, W be vector spaces over K.

8.1. A linear operator f on a K-vector space V is called an i n v o l u t i o n of V if $f^2 = \mathrm{id}_V$. Let $\mathrm{Inv}_K V$ (resp. $\mathrm{Proj}_K V$) denote the set of all involutions (resp. projections) of V. Suppose that $\mathrm{Char}\, K \neq 2$, i.e. $2 = 1_K + 1_K \neq 0$. Then the map $\gamma : \mathrm{Proj}_K V \to \mathrm{Inv}_K V$ defined by $p \mapsto \mathrm{id}_V - 2p$ is bijective. Further, for $p \in \mathrm{Proj}_K V$ show that

a). $\mathrm{im}\, p = \mathrm{Ker}\,(\mathrm{id} + \gamma(p))$ and $\mathrm{Ker}\, p = \mathrm{Ker}\,(\mathrm{id} - \gamma(p))$.

b). For an involution $f = \gamma(p)$ of V there is a direct sum decomposition :

$$V = V^- \oplus V^+$$

where $V^- := \{x \in V \mid f(x) = -x\} = \mathrm{im}\, p$ and $V^+ := \{x \in V \mid f(x) = x\} = \mathrm{Ker}\, p$.

8.2. Let $p_1, \ldots, p_n$ be distinct pairwise commuting projections of the K-vector space V. Then

a). The composition $p := p_1 \cdots p_n$ is a projection of V with $\mathrm{im}\, p = (\mathrm{im}\, p_1) \cap \cdots \cap (\mathrm{im}\, p_n)$ and $\mathrm{Ker}\, p = (\mathrm{Ker}\, p_1) + \cdots + (\mathrm{Ker}\, p_n)$.

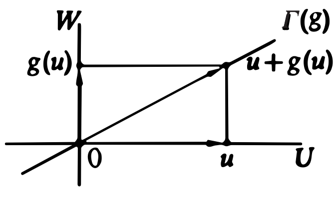
b). Let $q_1 := \mathrm{id}_V - p_1, \ldots, q_n := \mathrm{id}_V - p_n$ be the complementary projections. Then the projections $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ are pairwise commuting.

c). For $H = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_r$, let $p_H := p_{i_1} \cdots p_{i_r}$ and $q_H := q_{i_1} \cdots q_{i_r}$. Then $\mathrm{id}_V = \sum_{H \in \mathfrak{P}(\{1,2,\ldots,n\})} p_H q_H$, where H' denotes the complement $\{1, \ldots, n\} \setminus H$ of H in $\{1, \ldots, n\}$. (**Hint :** $\mathrm{id}_V = (p_1 + q_1) \cdots (p_n + q_n)$.)

d). V is the direct sum of the subspaces $U_H := \Big(\bigcap_{i \in H} \mathrm{Bild}\, p_i\Big) \cap \Big(\bigcap_{i \notin H} \mathrm{Kern}\, p_i\Big)$, $H \in \mathfrak{P}(\{1, \ldots, n\})$.
(**Hint :** For $H, L \subseteq \{1, \ldots, n\}$ with $H \neq L$, we have $p_H q_{H'} p_L q_{L'} = 0$.)

8.3. Suppose that the K-vector space V is the direct sum of the subspaces U and W.

a). For every linear map $g : U \to W$ the graph $\Gamma(g) := \{u + g(u) \mid u \in U\} \subseteq V$ of g is a complement of W in V.



b). The map $\mathrm{Hom}_K(U, W) \to \mathfrak{C}(W, V)$ defined by $g \mapsto \Gamma(g)$ is bijective, where denote the set of all complements of W in V.

c). Suppose that $\mathrm{Dim}_K U = \mathrm{Dim}_K W = n$. Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ be bases of U and W, respectively. Then $u_1 + w_1, \ldots, u_n + w_n$ is a basis of a complement of U as well as a complement of W in V.

d). Let V' be another K-vector space and let $f : V \to V'$ be a linear map of K-vector spaces such that $f|W : W \to \mathrm{im}\, f$ is bijective. Then there exists a unique K-linear map $g : U \to W$ such that $\mathrm{Ker}\, f = \Gamma(g) = \{u + w \mid u \in U, w = g(u)\}$. (**Remark :** In this case the equation $w = g(u)$ is called the s o l u t i o n o f t h e e q u a t i o n $f(x) = 0$, $x \in V$, f o r $w \in W$.)

8.4. Let V be a finite dimensional K-vector space and let $f : V \to V$ be an operator on V. Then

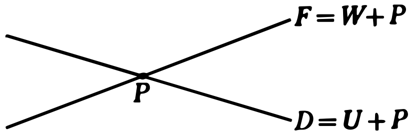
a). f is a projection of V if and only if there exists a basis $x_1, \ldots, x_n$ of V such that $f(x_i) = x_i$, $i = 1, \ldots, r$, and $f(x_i) = 0$, $i = r+1, \ldots, n$. (**Remark:** An analogous statement hold if V is not finite dimensional, formulate this and the prove!)

b). There exists an automorphism g and projections p, q of V such that $f = pg = gq$. (**Hint:** Extend a basis of $\operatorname{Ker} f$ to a basis of V. – In general, such a representation does not exists for operators on infinite dimensional K-vector spaces. Example!)

8.5. Let E be an affine space over the K-vector space V and let U, W be subspaces of V.

a). Any two affine subspaces F and F' of E which are parallel to U and W, respectively, intersects if and only if V is the sum of U and W.

b). Any two affine subspaces F and F' of E which are parallel to U and W, respectively, intersects exactly in a point if and only if V is the direct sum of U and W.



8.6. Let $f : V \to V''$ be a surjective K-linear map and let W be its kernel. Then the set of all complements U of W in V is an affine space over the K-vector space $\operatorname{Hom}_K(V'', W)$ with respect to the operation $(h, U) \longmapsto h + U := \{h(f(x)) + x \mid x \in U\}$, $h \in \operatorname{Hom}_K(V'', W)$.

On the other side one can see (simple) test-exercises; their solutions need not be submitted.

Test-Exercises

T8.1. **a).** The $\mathbb{K}$-vector space $\mathbb{K}^{\mathbb{R}}$ (resp. $\mathbb{K}^{\mathbb{K}}$) of the $\mathbb{K}$-valued functions on $\mathbb{R}$ (resp. $\mathbb{C}$) is the direct sums of the $\mathbb{K}$-subspaces $W_{\rm even}$ and $W_{\rm odd}$ of all even and all odd functions, respectively. (**Hint:** See exercise 2.3-d).)

b). Let V be a two dimensional K-vector space with basis x, y. The complements of the line Kx in V are the distinct lines of the form $K(ax+y)$, $a \in K$.

T8.2. Let $U_1, \ldots, U_n$ be subspaces of the K-vector space V. Then

a). The sum of the subspaces $U_1, \ldots, U_n$ is direct if and only if $(U_1 + \cdots + U_i) \cap U_{i+1} = 0$ for all $i = 1, \ldots, n-1$.

b). Assume that K has at least n elements, V is finite dimensional and all $U_1, \ldots, U_n$ have equal dimension. Then $U_1, \ldots, U_n$ have a common complement in V. (**Hint:** use the exercise 4.4)

c). Suppose that $U_1, \ldots, U_n$ are finite dimensional. Then $\operatorname{Dim}(U_1 + \cdots + U_n) \le \operatorname{Dim} U_1 + \cdots + \operatorname{Dim} U_n$. Moreover, the above inequality is and equality if and only if the sum of the U_i, $i = 1, \ldots, n$ is direct.

T8.3. Let U_i, $i \in I$ be a family of subspaces of the K-vector space V, let I_j, $j \in J$ be a partition of the indexed set I and let $W_j := \sum_{i \in I_j} U_i$, $j \in J$. The following statements are equivalent:

(1) The sum of the U_i, $i \in I$ is direct.
(2) For every $j \in J$ the sum of the U_i, $i \in I_j$, is direct and the sum of the W_j, $j \in J$, is direct.

T8.4. Let W be a complement of the subspace U in the vector space V. For every subspace V' of V with $U \subseteq V'$, the subspace $W \cap V'$ is a complement of U in V'.

T8.5. Suppose that the vector space V is the direct sum of its subspaces U and W. If $V = U' + W'$ with subspaces $U' \subseteq U$ and $W' \subseteq W$, then $U' = U$ and $W' = W$.

T8.6. Let $f : U \to V$ and $g : V \to W$ be homomorphisms of K-vector spaces. If gf is an isomorphism of U onto W, then V is the direct sum of $\operatorname{im} f$ and $\operatorname{Ker} g$.

T8.7. Let p be a projection and let f be an arbitrary operator on the K-vector space V.

a). p and f commute (i.e. $fp = pf$) if and only if the subspaces $\operatorname{im} p$ and $\operatorname{Ker} p$ are invariant under f, i.e. $f(\operatorname{im} p) \subseteq \operatorname{im} p$ and $f(\operatorname{Ker} p) \subseteq \operatorname{Ker} p$.

b). The subspace $\operatorname{im} p$ is invariant under f if and only if $fp = pfp$.

c). The subspace $\operatorname{Ker} p$ is invariant under f if and only if $pf = pfp$.

T8.8. In the situation of exercise 8.2, let $n = 2$. Then for two commuting projections p_1 and p_2 of V (by part d)) V is a direct sum of the K-subspaces $U_1 := \operatorname{im} p_1 \cap \operatorname{im} p_2$, $U_2 := \operatorname{im} p_1 \cap \operatorname{Ker} p_2$, $U_3 := \operatorname{Ker} p_1 \cap \operatorname{im} p_2$, $U_4 := \operatorname{Ker} p_1 \cap \operatorname{Ker} p_2$. For all 16 subsets $S \subseteq \{1, 2, 3, 4\}$ give (with the help of p_1 and p_2) the projection onto $\sum_{i \in S} U_i$ along $\sum_{i \notin S} U_i$.

T8.9. Let p and q be projections of the K-vector space V.

a). Show by an example that the composition pq can be a projection of V without the condition that p and q are commuting.

b). p and q have the same image if and only if $pq = q$ and $qp = p$.

c). Suppose that $\operatorname{Char} K \neq 2$, i.e. $2 = 1_K + 1_K \neq 0$ in K. Then $p+q$ is a projection of V if and only if $pq = qp = 0$. Moreover, in this case $\operatorname{im}(p+q) = \operatorname{im} p \oplus \operatorname{im} q$, and $\operatorname{Ker}(p+q) = (\operatorname{Ker} p) \cap (\operatorname{Ker} q)$.

d). Suppose that $\operatorname{Char} K = 2$. Then $p+q$ is a projection of V if and only if $pq = qp$. Moreover, in this case $\operatorname{im}(p+q) = (\operatorname{im} p \cap \operatorname{Ker} q) \oplus (\operatorname{im} q \cap \operatorname{Ker} p)$ and $\operatorname{Ker}(p+q) = (\operatorname{im} p \cap \operatorname{im} q) \oplus (\operatorname{Ker} p \cap \operatorname{Ker} q)$.

T8.10. Suppose that U and U' are two complements of the subspace W of the K-vector space V and p denote the projection of V onto U along W. Then $p|U' : U' \to U$ is an isomorphism.

T8.11. Let v_i, $i \in I$ be a basis of the finite dimensional K-vector space V and let U be a subspace of V. Then there exists a subset J of I such that the projection p_J onto $V_J := \sum_{i \in J} K v_i$ along $V_{I \setminus J} = \sum_{i \in I \setminus J} K v_i$ induces an isomorphism of U onto V_J. (**Remark:** This assertion is true even if I is not a finite set.)

T8.12. Let $f : V \to V'$ be a homomorphism of K-vector spaces. Then $W \subseteq V$ is a direct summand of $\operatorname{Ker} f$ in V if and only if f induces an isomorphism $f|W : W \to \operatorname{im} f$ of W onto $\operatorname{im} f$.

T8.13. Let V be a K-vector space.

a). Let $f_1 : U_1 \to V$, $f_2 : U_2 \to V$ be two surjective homomorphisms of K-vector spaces and let $f : U_1 \oplus U_2 \to V$ be the homomorphism defined by $f(x_1, x_2) := f_1(x_1) + f_2(x_2)$, $x_1 \in U_1$, $x_2 \in U_2$. Then $\operatorname{Ker} f_1 \oplus U_2 \cong \operatorname{Ker} f \cong U_1 \oplus \operatorname{Ker} f_2$.

b). Let $f : V \to V''$ be a surjective K-linear map, let $U \subseteq V$ be a K-subspace of V and let $f|U : U \to V''$ be the restriction of f to U. Then

(1) $f|U$ is injective if and only if $U \cap \operatorname{Ker} f = 0$.
(2) $f|U$ is surjective if and only if $U + \operatorname{Ker} f = V$.
(3) $f|U$ is an isomorphism if and only if $V = U \oplus \operatorname{Ker} f$, i.e. U is a complement of $\operatorname{Ker} f$ in V.

T8.14. Let K be a finite field with $\operatorname{card}(K) = q$ (note that $q = p^m$ for some $m \in \mathbb{N}^+$, where $p := \operatorname{Char} K$) and let V be an n-dimensional K-vector space.

a). For $n \in \mathbb{N}$, let $\alpha_q(n, r)$ be the number of linearly independent r-tuples $(x_1, \ldots, x_r) \in V^r$. For $1 \le r \le n$, show that $\alpha_q(n, r) = q^{(r-1)r/2} \prod_{i=n-r+1}^{n} (q^i - 1)$. In particular, $\alpha_q(n, r)$ depends only on q, n, r and does not depend on K and V. (**Hint:** Use induction on r.)

b). $\operatorname{card}(\operatorname{End}_K(V)) = q^{n^2}$ and $\operatorname{card}(\operatorname{Aut}_K(V)) = \alpha_q(n, n)$.

c). For $n \in \mathbb{N}$, let $\beta_q(n, r)$ be the number of r-dimensional K-subspaces of V. For $1 \le r \le n$, show that $\operatorname{Char} K$ does not divide $\beta_q(n, r)$ and $\beta_q(n, r) = \alpha_q(n, r)\alpha_q(r, r)^{-1}$. In particular, $\beta_q(n, r)$ depends only on q, n, r and does not depend on K and V.

d). The number of projections of V are $\sum_{r=0}^{n} \beta_q(n, r) q^{r(n-r)}$.

e). Let H be an elementary abelian p–group [1]) of order p^n, where p is a prime number. Compute the number of endomorphisms and automorphisms of H and the number of subgroups.

f). Let p be a prime numebr and let $n \in \mathbb{N}$. For $r \in \mathbb{Z}$, let $\begin{bmatrix} n \\ r \end{bmatrix}$ denote the number of subgroups of order p^r in an elementary abelain p–group of order p^n. This number is 0 for $r < 0$ and $r > n$; further,

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(p^n - 1)(p^{n-1} - 1) \cdots (p^{n-r+1} - 1)}{(p - 1)(p^2 - 1) \cdots (p^r - 1)}$$

for $0 \le r \le n$. (**Remark:** One can define these numbers by the above properties without any reference to the groups – and vector spaces. Note the similarity between these numbers and the binomial coefficients: $\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}$, and for $n \ge 1$, we have the recursion formula: $\begin{bmatrix} n \\ r \end{bmatrix} = p^r \begin{bmatrix} n-1 \\ r \end{bmatrix} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}$.)

g). In the set of subspaces of V which is ordered by the inclusion, the maximal number of elements which are not comparable is $\beta_q(n, [n/2])$.

[1]) The additive groups or the vector spaces over the field $\mathbf{K}_p = \mathbb{Z}/\mathbb{Z}p$ are called the e l e m e n t a r y a b e l i a n p - g r o u p s.