MA-219 Linear Algebra

9. Dual spaces

September 24, 2003 ; Submit solutions **before 11:00 AM ; October 03, 2003.**

Let K be a field and let V be a K-vector space.

9.1. Suppose that V is *not* finite dimensional and let v_i , $i \in I$ be a basis of V. Further, let v_i^*, $i \in I$ be the coordinate functions with respect to the basis v_i $i \in I$ and $W := \sum_{i\in I} K v_i^* \subseteq V^*$ be the subspace of V^* generated by $v_i^*, i \in I$. [1])

a). The linear form $\sum_{i\in I} a_i v_i \longmapsto \sum_{i\in I} a_i$ on V does not belong to W. In particular, $W \neq V^*$ and $v_i^*, i \in I$ *not* basis of V^*.

b). ${}^\circ W = 0$ and so $({}^\circ W)^\circ = V^* \neq W$.

c). The cannonical homomorphism $\sigma_V : V \to V^{**}$ is not surjective.

9.2. Let V be a K-vector space and let $f_1, \ldots, f_n \in V^*$ be linear forms on V. Let $f : V \to K^n$ be the homomorphism defined by $f(x) := \big(f_1(x), \ldots, f_n(x)\big)$. Then $\operatorname{Dim}(Kf_1 + \cdots + Kf_n) = \operatorname{Dim}(\operatorname{im} f)$. In particular, $f_1, \ldots, f_n$ are linearly independent if and only if the homomorphism f is surjective.

9.3. Suppose that V is a finite dimensional. Then

a). For every basis f_i, $i \in I$ of V^*, there exists a (unique) basis v_i, $i \in I$ of V such that $f_i = v_i^*$, $i \in I$.

b). $\operatorname{Dim} U = \operatorname{Codim}(U^\circ, V^*)$ for every subspace $U \subseteq V$. (**Remark:** It is enough to assume that U is finite dimensional.)

c). For subspaces $U_1, U_2 \subseteq V$ resp. $W_1, W_2 \subseteq V^*$, show that

$$(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ\,, \qquad (U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ\,,$$
$${}^\circ(W_1 + W_2) = {}^\circ W_1 \cap {}^\circ W_2\,, \qquad {}^\circ(W_1 \cap W_2) = {}^\circ W_1 + {}^\circ W_2\,.$$

9.4. Let $r \in \mathbb{N}$. The maps $W \mapsto {}^\circ W$ and $U \mapsto U^\circ$ are inverses of each other on the set of all r-dimensional subspaces W of V^* and the set of all r-codimensional subspaces U of V. (**Remark:** A subspace $U \subseteq V$ is called r-c o d i m e n s i o n a l in V if one (and hence every) of the complement of U in V is r-dimensional. – the map $U \mapsto U^\circ$ from the set of all r-dimensional subspace U of V into the set of all r-codimensional subspaces of V^* is injective, see exercise 9.??. But not surjective in the case when V is not finite dimensional.)

9.5. Let $f : V \to W$ be a homomorphism of K-vector spaces.

a). The K-linear map f is injective resp. surjective resp. bijective resp. 0 if and only if the dual map $f^* : W^* \to V^*$ is surjective resp. injective resp. bijective resp. 0.

b). The kernel of the dual map $f^* : W^* \to V^*$ is is the space of all linear forms $g : W \to K$ on W, which vanish on the im f and so $\operatorname{Kern} f^* = (\operatorname{Bild} f)^\circ$. The image of f^* is the space of all linear forms $V \to K$, which vanish on the Ker f and so $\operatorname{im} f^* = (\operatorname{Ker} f)^\circ$.

9.6. Let $x_1, \ldots, x_n$ be all non-zero vectors in a K-vector space V over a field K with at least n elements. Then there exists a hyperplane H in V such that the vectors $x_i \notin H$ for all $i = 1, \ldots, n$. (**Hint:** There exist a linear form $f : V \to K$ such that $f(x_i) \neq 0$ for all $i = 1, \ldots, n$.)

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

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[1]) Consider in particular, the concrete situation $V := K^{(I)}$, $v_i := e_i$, $i \in I$ with $V^* \cong K^I$, $W \cong K^{(I)} \subset K^I$.

Test-Exercises

T9.1. For a subspace U of V, the following statements are equivalent:

(1) $U \neq V$ and there exists a $v \in V$ such that $V = U + Kv$.
(1′) There exists a $v \in V$, $v \neq 0$ such that $V = U \oplus Kv$.
(2) There exists a linear form $f \neq 0$ on V such that $U = \text{Kern } f$. (**Remark:** The subspaces U with these properties are called h y p e r p l a n e s in V.)

T9.2. a). If V^* is finite dimensional, then V is finite dimensional.

b). Let $v_1, \ldots, v_n$ be a basis of V. For $a_1, \ldots, a_n \in K$, find a basis of the kernel of the linear form $a_1 v_1^* + \cdots + a_n v_n^*$.

T9.3. Let K be a subfield of the field L.

a). A family $f_i \in K^D$, $i \in I$ of K-valued functions on D is linearly independent over K if and only if the family f_i, $i \in I$ as a family of L-valued functions on D is linearly independent over L. Further, $\text{Dim}_K \left(\sum_{i \in I} K f_i \right) = \text{Dim}_L \left(\sum_{i \in I} L f_i \right)$ for an arbitrary family $f_i \in K^D$, $i \in I$.

b). Let W be a K-subspace of the K-vector space K^D and $L \cdot W$ be the L-subspace of the L-vector space L^D generated by W. Then $K^D \cap L \cdot W = W$. (**Hint:** Let $f \in K^D \cap LW$, but $f \notin W$. Then f can be expressed as $f = c_1 f_1 + \cdots + c_r f_r$ with $c_1, \ldots, c_r \in L$ and linear independent functions $f_1, \ldots, f_r \in W$. Then $f, f_1, \ldots, f_r$ are linearly independent over K, but are linearly dependent over L, a contradiction!)

9.7. (L i n e a r I n d e p e n d e n c e o f f u n c t i o n s) Let D be an arbitrary set and let $f_1, \ldots, f_n \in K^D$ be K-valued functions on D. Let W denote the subspace of K^D generated by these functions.

a). The following statements are equivalent:

(1) The functions $f_1, \ldots, f_n$ are linearly independent in K^D.
(1′) $\text{Dim}_K W = n$.
(2) The image of f is a generating system of K^n.
(2′) There exist elements $t_1, \ldots, t_n \in D$ such that the images $f(t_i) = \big(f_1(t_i), \ldots, f_n(t_i)\big)$, $i = 1, \ldots, n$, is a generating system (i.e. a basis) of K^n.
(3) There exists a subset $E \subseteq D$ with $|E| = n$ such that the restrictions $f_1|E, \ldots, f_n|E$ are linearly independent in K^E (and hence form a basis of K^E).
(3′) There exist elements $t_1, \ldots, t_n \in D$ such that the n-tuples $\big(f_j(t_1), \ldots, f_j(t_n)\big)$, $j = 1, \ldots, n$, are linearly independent in K^n (and hence form a basis of K^n).
(4) There exist function $g_1, \ldots, g_n \in W$ and elements $t_1, \ldots t_n \in D$ such that $g_j(t_i) = \delta_{ij}$ for $1 \leq i, j \leq n$.

b). Let $f : D \to K^n$ be the map defined by $t \mapsto f(t) := \big(f_1(t), \ldots, f_n(t)\big)$. Then Dim W is equal to the dimension of the subspaces of K^n generated by the image im f of f.

T9.4. ($\mathbb{C}$- a n t i - l i n e a r f o r m s) Let V be a $\mathbb{C}$-Vektorraum. A $\mathbb{C}$-anti-linear map $V \to \mathbb{C}$ is called a $\mathbb{C}$- a n t i - l i n e a r f o r m on V. The $\mathbb{C}$-vector space of the $\mathbb{C}$-anti-linear forms on V is denoted by $\overline{V}^*$.

a). $f : V \to \mathbb{C}$ is linear over $\mathbb{C}$ if and only if $\overline{f} : V \to \mathbb{C}$ ($x \mapsto \overline{f(x)}$) is $\mathbb{C}$-anti-linear. The linear forms $f_i \in V^*$, $i \in I$ form a $\mathbb{C}$-basis of V^* if and only if the $\mathbb{C}$-anti-linear forms $\overline{f}_i$, $i \in I$ form a $\mathbb{C}$-basis of $\overline{V}^*$.

b). If v_i, $i \in I$ is a finite $\mathbb{C}$-basis of V, then $\overline{v_i^*}$, $i \in I$ is a $\mathbb{C}$-basis of $\overline{V}^*$. In particular, $\text{Dim}_{\mathbb{C}} V = \text{Dim}_{\mathbb{C}} V^* = \text{Dim}_{\mathbb{C}} \overline{V}^*$ for every finite dimensional $\mathbb{C}$-vector spaces V.

c). $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^* \oplus \overline{V}^*$ $(\subseteq \mathbb{C}^V)$.

T9.5. Let $K \subseteq L$ be a field extension and let V be a L-vector space (and hence it is also a K-vector space by the restriction of scalars). Further, let $\sigma : L \to K$ be a K-linear form $\neq 0$. (**Remark:** Such a function is also called a g e n e r a l i s e d t r a c e f u n c t i o n. In the case $\mathbb{R} \subseteq \mathbb{C}$ one may choose $\sigma := \text{Re}$.)$\text{Hom}_K(V, K)$ is L-vector space with scalar multiplication $(bf)(x) := f(bx)$ for $b \in L$, $x \in V$ and $f \in \text{Hom}_K(V, K)$.

a). Let $[L : K] < \infty$. Then the map $\text{Hom}_L(V, L) \xrightarrow{\sim} \text{Hom}_K(V, K)$ defined by $f \mapsto \sigma \circ f$ is an isomorphism of L-vector spaces. (**Hint:** With the help of a L-basis of V one can reduce to the case $V = L$. In this case use a dimension-argument. In the case $\mathbb{R} \subseteq \mathbb{C}$ and $\sigma := \text{Re}$ the map $g \longmapsto \big(x \mapsto g(x) - \mathrm{i}\, g(\mathrm{i}x)\big)$ is the inverse map.)

b). If $[L : K] < \infty$. Then every K-subspace $U \subseteq V$ with $\text{Codim}_K(U, V) = r \in \mathbb{N}$ is contain a L-subspace U' with $\text{Codim}_L(U', V) \leq r$. (See exercise 9.4.)

c). There exists a $\mathbb{Q}$-hyperplane H in $\mathbb{R}^2$ such that H donot contain any $\mathbb{R}$-hyperplane in $\mathbb{R}^2$.