MA-219 Linear Algebra

10. Quotient spaces – Exact sequences

September 24, 2003 ; Submit solutions **before 11:00 AM ; October 03, 2003.**

Let K denote a field and let V denote a K-vector space.

10.1. Let $n \in \mathbb{N}$. A subspace U of the K-vector space V has codimension n if and only there exists n linearly independent linear forms $f_1, \ldots, f_n$ on V such that $U = \bigcap_{i=1}^n \operatorname{Kern} f_i$.

10.2. Let $U_1, \ldots, U_n$ be finite codimensional subspaces of V and let $U := \bigcap_{i=1}^n U_i$. Further, let $U_i' := \bigcap_{j \neq i} U_j$, $i = 1, \ldots, n$.

a). U is finite codimensional and $\operatorname{Codim}(U, V) \leq \sum_{i=1}^n \operatorname{Codim}(U_i, V)$.

b). The following statements are equivalent :

(1) The inequality in the part a) is an equality.
(2) The canonical homomorphism $V/U \to \bigoplus_{i=1}^n V/U_i$ is an isomorphism.
(3) $U_i + U_i' = V$ for all $i = 1, \ldots, n$.
(4) Es ist $U_1' + \cdots + U_n' = V$.
(5) The sum of the subspaces U_i° , $i = 1, \ldots, n$, in V^* is direct.

c). Let $f : V \to W$ be a linear map of finite dimensional K-vector spaces. Then

$$\operatorname{Dim} \ker f - \operatorname{Dim} \operatorname{Coker} f = \operatorname{Dim} V - \operatorname{Dim} W .$$

10.3. Let f be an operator on V. Then the following statements are equivalent:

(1) f induces an automorphism of $\operatorname{im} f$.
(2) f induces an automorphism of $V/\ker f$.
(3) $V = \operatorname{im} f \oplus \operatorname{Kern} f$.
(4) $\ker f$ has a f-invariant complement W such that $f|W$ is an automorphismus of W. (**Remark :** The subspace W in (4) must be $\operatorname{im} f$.)

10.4. (Euler-Poincaré-Characteristic) Let

$$V_\bullet : \quad 0 \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$$

be a complex of finite dimensional K-vector spaces and let $H_0, H_1, \ldots, H_{n-1}, H_n$ be the corresponding homology spaces. Then

$$\sum_{i=0}^n (-1)^i \operatorname{Dim}_K H_i = \sum_{i=0}^n (-1)^i \operatorname{Dim}_K V_i .$$

(**Remark :** The alternating sum on the left hand side is called the Euler-Poincaré-Characteristic of the complex $V_\bullet$ and is usually denoted by $\chi(V_\bullet)$. This can be defined if the homology spaces H_i , $i = 0, \ldots, n$ are finite dimensional .)

10.5. (Index of a linear map) If the kernel and the cokernel of a K-linear map $f : V \to W$ are finite dimensional, then we say that f has a index and $\operatorname{Ind} f := \operatorname{Dim}_K \operatorname{Ker} f - \operatorname{Dim}_K \operatorname{Coker} f$ is called the index of f . (**Remark :** In this case $\operatorname{Ind} f$ is also the Euler-Poincaré-Characteristic of th complex $0 \longrightarrow V^0 \xrightarrow{f^0} V^1 \longrightarrow 0$, where $V^0 := V$, $V^1 := W$, $f^0 := f$.)

a). If V and W are finite dimensional, then $\operatorname{Ind} f = \operatorname{Dim}_K V - \operatorname{Dim}_K W$.

b). Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_n & \longrightarrow & \cdots & \longrightarrow & V_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & W_n & \longrightarrow & \cdots & \longrightarrow & W_0 & \longrightarrow & 0 \end{array}$$

be a commutative diagramm of K-vector spaces and K-linear maps with exact rows. Then all but at most of the linear maps $f_0, \ldots, f_n$ have an index, then all these maps have an index and $\sum_{i=0}^{n} \operatorname{Ind} f_i = 0$. (**Hint:** Prove the assertion by induction on n. In the case $n = 2$, use the snake-lemma from exercise T10.4.)

c). If $f : V \to W$ and $g : W \to X$ have index, then the composite $gf : V \to X$ also has an index and $\operatorname{Ind} gf = \operatorname{Ind} g + \operatorname{Ind} f$. (**Hint:** Consider the following commutative diagramm:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & V & \xrightarrow{f} & W & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \overline{g}} & & \\ 0 & \longrightarrow & \operatorname{Ker} gf & \longrightarrow & V & \xrightarrow{gf} & X & \longrightarrow & \operatorname{Coker} gf & \longrightarrow & 0 \end{array}$$
.)

d). If $f : V \to W$ has an index and if $g : V \to W$ has a finite rank, then $f + g$ has an index and $\operatorname{Ind}(f + g) = \operatorname{Ind} f$. (**Hint:** Let $U := \operatorname{im} g$ and $(f, g)(x) := \big(f(x), g(x)\big)$ and consider the commutative diagramms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & V & \xrightarrow{\mathrm{id}} & V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle (f,g)} & & \downarrow{\scriptstyle f+g} & & \\ 0 & \longrightarrow & U & \longrightarrow & W \oplus U & \longrightarrow & W & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & V & \xrightarrow{\mathrm{id}} & V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle (f,g)} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & U & \longrightarrow & W \oplus U & \longrightarrow & W & \longrightarrow & 0 \end{array}$$
.)

e). The map $f : V \to W$ has an index if and only if the dual map $f^* : W^* \to V^*$ has an index and in this case $\operatorname{Ind} f^* = -\operatorname{Ind} f$.

10.6. Let $V' \to V \to V''$ be a complex of K-vector spaces with the homology space H and let X be an another K-vector space. Then the homology spaces of the complexes

$$\operatorname{Hom}_K(V'', X) \longrightarrow \operatorname{Hom}_K(V, X) \longrightarrow \operatorname{Hom}_K(V', X),$$

$$\operatorname{Hom}_K(X, V') \longrightarrow \operatorname{Hom}_K(X, V) \longrightarrow \operatorname{Hom}_K(X, V'')$$

are canonically isomorphic to $\operatorname{Hom}_K(H, X)$ resp. to $\operatorname{Hom}_K(X, H)$. In particular, if $X \neq 0$, then from the exactness of one the Hom-Sequence, the exactness of the other follows.

On the other side one can see (simple) test-exercises; their solutions need not be submitted.

Test-Exercises

T10.1. Let U, W be subspaces of the K-vector space V with $U \subseteq W$. If W' is a complement of W in V, then $(U + W')/U$ is a complement of W/U in V/U, which is isomorphic to W'

T10.2. a). Let $\varphi : V \to V'$ be a homomorphism of K-vector spaces and let U resp. U' be subspaces of V resp. V' with $\varphi(U) \subseteq U'$. Then φ induces a homomorphism $\overline{\varphi} : V/U \to V'/U'$ such that $\overline{\varphi}(x + U) = \varphi(x) + U'$, i.e. the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\ \varphi\ } & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ V/U & \overset{\overline{\varphi}}{\dashrightarrow} & V'/U' \end{array}$$

b). (Noether's isomorphism theorem) Let U and W be subspaces of K-vector space V. The natural injective map $\iota : U \to U + W$ induces an isomorphism $\bar{\iota} : U/(U \cap W) \cong (U + W)/W$. The following diagram is commutative.

$$\begin{array}{ccc} U & \xrightarrow{\ \iota\ } & U + W \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U/(U \cap W) & \overset{\bar{\iota}}{\dashrightarrow} & (U + W)/W \end{array}$$

c). Let U and W be subspaces of the K-vector space V with $W \subseteq U$. Then the indentity map of V induces a homomorphism $V/W \to V/U$ and futher induces an isomorphism $(V/W)/(U/W) \cong V/U$. The following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\ \mathrm{id}_V\ } & V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ V/W & \overset{\overline{\mathrm{id}}_V}{\dashrightarrow} & V/U \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\overline{\overline{\mathrm{id}}}_V} & \\ (V/W)/(U/W) & & \end{array}$$

T10.3. Let H and F be subgroups of the abelain group G. Then the sequences

$$0 \longrightarrow H \cap F \xrightarrow{f} H \oplus F \xrightarrow{g} H + F \longrightarrow 0$$

with $f(x) = (x, -x)$ und $g(y, z) = y + z$ and

$$0 \longrightarrow G/(H \cap F) \xrightarrow{h} (G/H) \oplus (G/F) \xrightarrow{k} G/(H + F) \longrightarrow 0$$

with $h(\overline{x}) = (\overline{x}, -\overline{x})$ und $k(\overline{y}, \overline{z}) = \overline{y + z}$ are exact.

T10.4. (Snake-lemma) Suppose that the diagram

$$\begin{array}{ccccccc} & & G' & \xrightarrow{g'} & G & \xrightarrow{g} & G'' & \longrightarrow & 0 \\ & & {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h''}\downarrow & & \\ 0 & \longrightarrow & F' & \xrightarrow{f'} & F & \xrightarrow{f} & F'' & & \end{array}$$

of abelain groups and group homomorphisms is commutative and its rows are exact. Then the complexes

$$\operatorname{Ker} h' \xrightarrow{g'} \operatorname{Ker} h \xrightarrow{g} \operatorname{Ker} h'' \quad \text{and} \quad \operatorname{Coker} h' \xrightarrow{\overline{f'}} \operatorname{Coker} h \xrightarrow{\overline{f}} \operatorname{Coker} h'',$$

are exact. Further (more important), there is a canonical homomorphism $\delta : \operatorname{Kern} h'' \longrightarrow \operatorname{Kokern} h'$ which connects both the exact sequences to a long exact seqeunce called the exact Ker-Coker-sequence [1]).

$$\operatorname{Ker} h' \xrightarrow{g'} \operatorname{Ker} h \xrightarrow{g} \operatorname{Ker} h'' \overset{\delta}{\dashrightarrow} \operatorname{Coker} h' \xrightarrow{\overline{f'}} \operatorname{Coker} h \xrightarrow{\overline{f}} \operatorname{Coker} h'',$$

[1]) This exact sequence explains the name “Snake-lemma”.

The homomorphism δ is called the connecting homomorphismu. (**Proof** Define δ as follows: Let $x'' \in \operatorname{Kern} h''$. Since g is surjective, there exists $x \in G$ with $g(x) = x''$. Then $f h(x) = h'' g(x) = h''(x'') = 0$, and so $h(x) \in \operatorname{Ker} f = \operatorname{im} f'$ and $h(x) = f'(y')$ with (unique) $y' \in F'$. Define $\delta(x'') := \overline{y'} \in \operatorname{Kokern} h' = F'/\operatorname{Bild} h'$. The image $\delta(x'')$ does not depend on the choice of the inverse image x of x''. For, if $g(\tilde{x}) = x''$, then $x - \tilde{x} \in \operatorname{Ker} g = \operatorname{im} g'$, i.e. $x - \tilde{x} = g'(x')$ and for $\tilde{y}' \in F'$ with $h(\tilde{x}) = f'(\tilde{y}')$ gilt $y' - \tilde{y}' = h'(x')$, therefore $\overline{y'} = \overline{\tilde{y}'}$ in $F'/\operatorname{im} h'$.

It can be easily checked that δ is a homomorphism. Further, the exactness of the above long seqeunce at the places $\operatorname{Ker} h''$ and $\operatorname{Coker} h'$ can be easily verified by diagram chasing (similar to the above proof of independence and the proof of exactness at the other places.)

If g' is injective resp. f is surjective, then $\operatorname{Ker} h' \longrightarrow \operatorname{Ker} h$ is injective resp. $\operatorname{Coker} h \longrightarrow \operatorname{Coker} h''$ is surjective.

T10.5. (Five-Lemma) Suppose that the following diagram of abelian groups and group homomorphisms

$$\begin{array}{ccccccccc} G_5 & \longrightarrow & G_4 & \longrightarrow & G_3 & \longrightarrow & G_2 & \longrightarrow & G_1 \\ {\scriptstyle h_5}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_1}\downarrow \\ F_5 & \longrightarrow & F_4 & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 \end{array}$$

has exact rows. Then :

a). If h_2 and h_4 are injective and h_5 is surjective, then h_3 injective.

b). If h_2 and h_4 are surjective and h_1 is injective, then h_3 is surjective.

c). If h_1, h_2, h_4, h_5 are bijective, then h_3 is bijektive.

T10.6. Let $0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow 0$ be a complex of finite abelian groups with the homology groups $H_0, \ldots, H_n$. Then $\prod_{i=0}^n |H_i|^{(-1)^i} = \prod_{i=0}^n |G_i|^{(-1)^i}$. (**Hint:** Use the exercise 10.4.)

T10.7. (Herbrand-Quotients) If the kernel and cokernel of a homomorphism $h : G \to F$ of abelian groups are finite, then we say that h has a Herbrand-Quotients and define this by

$$\mathrm{q}(h) := |\operatorname{Ker} h| / |\operatorname{Koker} h| \, .$$

a). If G and F are finite then $\mathrm{q}(h) = |G|/|F|$.

b). Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_n & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & 0 \\ & & {\scriptstyle h_n}\downarrow & & & & {\scriptstyle h_0}\downarrow & & \\ 0 & \longrightarrow & F_n & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & 0 \end{array}$$

be a commutative diagramm of abelian groups and group homomorphisms. If all but at most one of homomorphisms $h_1, \ldots, h_n$ have a Herbrand-Quotients, then all have Herbrand-Quotients and

$$\prod_{i=0}^{n} \mathrm{q}(h_i)^{(-1)^i} = 1.$$

c). If $h : G \to F$ and $j : F \to E$ have a Herbrand-Quotients then the composite homomorphismus $jh : G \to E$ also have Herbrand-Quotient and $\mathrm{q}(jh) = \mathrm{q}(j)\,\mathrm{q}(h)$.

d). If $h : G \to F$ has a Herbrand-Quotient and if $j : G \to F$ is a homomorphism with a finite image, then $h + j$ has a Herbrand-Quotient and $\mathrm{q}(h + j) = \mathrm{q}(h)$.