MA-219 Linear Algebra

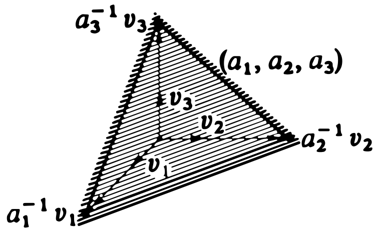
12. Affine maps

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

12.1. (Affine hyperplanes) Let V be a non-zero K-vector space and let v_i, $i \in I$, be a basis of V. The map $h \mapsto h^{-1}(1)$ is a bijective map from the set of all non-zero K-linear forms $h : V \to K$ onto the set of all affine hyperplanes in V which do not pass through the 0. For a hyperplane H in V with $0 \notin H$ the corresponding linear form is denoted by h_H . Then by definition $H = h_H^{-1}(1)$.

a). Two hyperplanes H, H' in V, which donot pass through 0 are parallel if and only if the corresponding linear forms h_H resp. $h_{H'}$ differ by a factor $\lambda \in K^\times$. The hyperplane passing through 0 and parallel to H is $\operatorname{Kern} h_H = h_H^{-1}(0)$. The hyperplane *directions* in V correspond to the unique one dimensional K-subspaces in the dual space V^*.

b). The map $V^* \to K^I$ defined by $h \mapsto (h(v_i))_{i\in I}$ is a vector space isomorphism. For an affine hyperplane H in V with $0 \notin H$ the values $a_i := h_H(v_i)$, $i \in I$, are called (particularly in the *Crystallography*) the Miller's indices of H and the tuple $(a_i) \in K^I$ is called the (hyperplane-) symbol of H with respect to the basis v_i , $i \in I$. If I is finite then $h_H = \sum_i a_i v_i^*$ if (a_i) is the symbol of H. Two affine hyperplanes which donot pass through 0, are parallel if and only if their symbols differ by a factor $\lambda \in K^\times$. If $(a_i) \in K^I$ is the symbol of the hyperplane H, then the i-th coordinate axis Kv_i, $i \in I$ intersects with the hyperplane H if and only if $a_i \neq 0$. In this case $a_i^{-1} v_i$ is the point of intersection of H with Kv_i . (*The Miller's indices a_i , $i \in I$, of H are therefore the inverses of the intercepts of H on the coordinate axes.* Such an intercept is equal to 0 $(= 1/\infty)$, in the case H donot intersect with axis, i.e. H is parallel to this axis.



12.2. (Affine functions) Let E be an affine space over the K-vector space V.

a). The set of all affine functions $E \to K$ is a K-subspace of the vector space of all K- valued functions on E. If E is finite dimensional, then the dimension of the space of affine functions on E is equal to $1 + \operatorname{Dim} E$.

b). If H is a hyperplane in E, then there exists a non-constant affine function $f : E \to K$ with $H = f^{-1}(0)$. Further, f is uniquely determined by H, upto a factor $\lambda \in \mathbb{K}^\times$. (**Remark:** Therefore the hyperplanes in E can be identified with a uniquely determined non-constant affine functions $E \to K$, where two such functions are identified if they differ by a factor $\lambda \in K^\times$.)

c). Let v_i , $i \in I$ be a basis of V and let $E = V$. For an affine hyperplane H in V with $0 \notin H$, let $f : V \to K$ be an affine function with $H = f^{-1}(0)$. Further, let $f_0 : V \to K$ be the linear part of f and $b := f(0)\ (\neq 0)$. Then $H = h_H^{-1}(1)$ with $h_H := -b^{-1} f_0$, see exercise T10.??. The Miller's indices of H with respect to the basis v_i , $i \in I$ are the numbers $-b^{-1} f_0(v_i)$, $i \in I$.

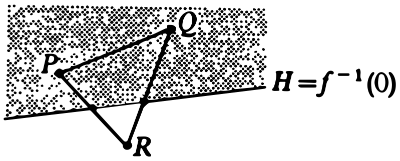
12.3. (Half-spaces) Let E be an real affine space over the non-zero $\mathbb{R}$-vector space V and let H be an affine hyperplane in E. Suppose that $H = f^{-1}(0)$, where F is a non-constant affine function on E, see exercise T10.??. Then the sets

$$\{P \in E \mid f(P) \leq 0\} \qquad \text{resp.} \qquad \{P \in E \mid f(P) \geq 0\}$$

are called the closed and the sets

$$\{P \in E \mid f(P) < 0\} \qquad \text{resp.} \qquad \{P \in E \mid f(P) > 0\}$$

are called the open affine half-spaces of E with respect to H. The half-space with respect to the affine hyperplane $H \subseteq E$ are convex subsets of E. Two points $P, Q \in E$ belong to the same open half-space with respect to the hyperplane H if and only if the line-segment $[P, Q]$ donot intersect with the hyperplane H.



12.4. (Parallel projections) Let E be an affine space over the vector space V. An affine map $p : E \to E$ is called the ((diagonal) parallel-)projection if $p^2 = p$. The linear part of a projection of E is a (linear) projection of V. Let p be a projection of E with linear part p_0.

a). For every point $Q \in \operatorname{im} p$, Q is the only point of intersection of the image $\operatorname{im} p$ of p with the fiber $p^{-1}(Q)$ of p over Q, i.e. $\operatorname{im} p \cap p^{-1}(Q) = \{Q\}$ and E is the join-space of $\operatorname{im} p$ and $p^{-1}(Q)$, i.e. $E = \operatorname{im} p \vee p^{-1}(Q)$. If $R \in E$ is an arbitrary point, then $p(R)$ is the point of intersection of $\operatorname{im} p$ and the fibre parallel to $p^{-1}(Q)$ through R.

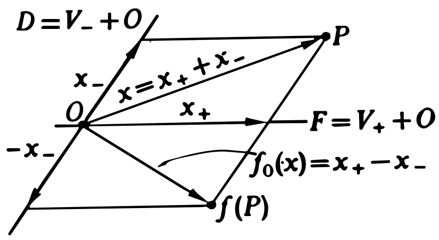
b). Conversely, given two affine subspaces D and F of E such that they intersect in exactly one point and their join-space is the whole E, then there exist a unique parallel projection $P_{D,F}$ such that its image is D and the fibres are parallel to F. Such a projection $P_{D,F}$ is also called the projection onto D along F. – For which translations $t : E \to E$ of E, the affine maps $t \circ p$ resp. $p \circ t$ are projections of E ?

12.5. (Reflections and Glide-reflections) Let E be an affine space over the K- vector space V. Suppose that $2 = 1_K + 1_K \neq 0$ in K, i.e. Char $K \neq 2$. An affine map $f : E \to E$ is called an affine involution or reflection of E if $f^2 = \mathrm{id}_E$. The linear part f_0 of an affine involution f is a linear involution and therefore there is the corresponding decomposition $V = V^+ \oplus V^-$ of V in the sense of exercise ???. Let $f : E \to E$ be an affine involution.

a). The subspace F of the fixed points of f is non-empty. (**Hint:** In fact for every point $P \in E$ the midpoint $\frac{1}{2}P + \frac{1}{2}f(P)$ of P and $f(P)$ is a fixed point of f. Further, $F = V^+ + P$ for every point $P \in F$. This affine subspace F is called the mirror of f.)

b). The map f induces a point-reflection (see exercise T10.??) on every affine subspace D of E which is parallel to the subspace V^- and the (unique) point of intersections of D and F is the center of this point-reflection.

c). Conversely, suppose that D and F are two affine subspaces of E which intersect in exactly one point and the join-space of D and F is E. Then there exists an unique affine involution of E such that its fixed point set is F and on every affine subspace of E which is parallel to D, the map f induces a point-reflection. (**Remark:** This involution is called the (diagonal) reflection on F along D.)



d). In the situation of c), let $D = U + O$ and $F = W + O$. Further, let F be a hyperplane and hence D a line, therefore $U = Kx$ with a vector $x \in V \setminus W$. If h is the linear form on V with $\operatorname{Ker} h = W$ and $h(x) = 1$, then the reflecion $S_{F,D}$ on F along D can be described by the equation

$$S_{F,D}(P) = -2\,h(\overrightarrow{OP})\,x + P\,.$$

e). For which translations $t : E \to E$, the affine map $t \circ f$ resp. $f \circ t$ is an affine involution? (**Hint:** Since f is an affine involution, it has a fixed point $O \in E$. We may choose O as origin in E (see the commuttative diagramm in the hint of T12.7) and assume that $E = V$ and $f = f_0$.)

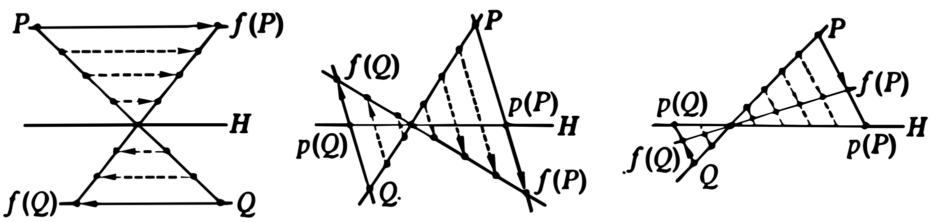
f). The affinity $f : E \to E$ is called a (diagonal) glide reflection if its linear part $f_0 : V \to V$ is an involution. Every reflection is a glide reflection. Let $f : E \to E$ be a glide reflection and $V = V^+ \oplus V^-$ is the decomposition of V with respect to f_0 as above. Then $f = \widetilde{f_0} \circ t = t \circ \widetilde{f}$ with a uniquely determined reflection $\widetilde{f}$ and a uniquely determined translation t in the direction V^+.

12.6. (Shearings and Dilatations) Let E be an affine space over the K-vector space V of dimension ≥ 2. An affinity $f : E \to E$ is called a pseudo-reflection if the fixed point set of f is an affine hyperplane in E.

a). The affinity $f : E \to E$ is a pseudo-reflection if and only if it has a fixed point and its linear part f_0 has the property that $\operatorname{Rank}(f_0 - \mathrm{id}_V) = 1$. The hyperplane of the fixed points is called the reflection plane of f.

Now, let f be a pseudo-reflection of E with reflection-plane H.

b). For two points $P, Q \in E \setminus H$, the line-segments $P\,f(P)$ and $Q\,f(Q)$ are parallel. (**Remark:** The direction of this line is called the reflection-direction. If the reflection-direction and the reflection-plane are parallel, then f is called a shearing or transvection, otherwise f is called a dilatation. Reflection on hyperplanes in the sense of exercise 10.?? are examples of dilatations.)



Shearing Dilatation with $\lambda(f)<0$ Dilatation with $\lambda(f)>0$

c). Let f be a dilatation. For a point $P \in E \setminus H$, let $p(P)$ be the point of intersection of the line-segment $P\,f(P)$ with the hyperplane H. Then the ratio $\big(p(P), f(P)\big) : \big(p(P), P\big)$ has the same value for all $P \in E \setminus H$. This value is called the magnification ratio of the dilatation and is denoted by $\lambda(f)$. On every line parallel to the reflection-direction f induces a magnification (see exercise T10.??) with magnification ratio $\lambda(f)$.

d). The transvections with reflection-hyperplane H together with the identity form a subgroup of $\mathrm{A}(E)$, which is isomorphic to the group of translations of H. (**Hint:** In fact an isomprphism is given by $f \mapsto \overrightarrow{P_0 f(P_0)}$, where $P_0 \in E \setminus H$ is fixed.)

e). The dilatations with fixed reflection-hyperplane and a fixed reflection-direction together with the identity form a subgroup of $\mathrm{A}(E)$, which is isomorphic to the multiplicative group $K^\times$ of K. (**Hint:** In fact an isomorphism is given by $f \mapsto \lambda(f)$, where $\lambda(\mathrm{id}) := 1$.)

f). For which translations $t : E \to E$ of E, the affine map $t \circ f$ resp. $f \circ t$ is a pseudo-reflection? (**Hint:** Since f is a pseudo-reflection, it has a fixed point $O \in E$. We may choose O as origin in E (see the commuttative diagramm in the hint of T12.7) and assume that $E = V$ and $f = f_0$.)

12.7. Let E be an n-dimensional affine space. Then every affinity g of E is a product of a dilatation d with at most $2n$ shearings. (**Hint:** If the dimension of the fixed point set of g is $k \in \{-1, 0, \ldots, n-2\}$, then there exists (see T12.8) an affinity $h : E \to E$, which is a shearing or the product of two shearings such that the dimension of the fixed point set of $h \circ g$ is at least $k+1$. – The magnification ratio of the dialatation d in the above representation of g is uniquely determined and is equal to the determinant of the linear part of g. Further, the dilatation is not necessary if and only if this determinant is 1. – An another proof of this assertion is a consequence of the representation of matrices by elementary matrices, see ???.)

12.8. Let A be a finite subset of m points in an affine space E over the K-vector space V. Assume that $m \cdot 1_K \neq 0$. For every affinity of E, which maps A onto itself, the center of mass S of the points of A is a fixed point. The group of affinities of E, which maps A onto itself is therefore a subgroup of the group of affinities $\mathrm{A}_S(E)$ $(\cong \mathrm{GL}(V))$ of E with fixed point S. (**Hint:** See exercise T12.??)

12.9. Let E be a K-affine space over V and G be a finite subgroup of the affine group of E with n elements.

a). If $n \cdot 1_K \neq 0$, then there exists a point $O \in E$ with $g(O) = O$ for all $g \in G$. (**Hint:** For O one can take the point $\sum_{g \in G} \frac{1}{n} g(P)$, where P is an arbitrary point in E. – G is therefore a subgroup of $\mathrm{A}_O(E) \cong \mathrm{GL}(V)$.)

b). Assume that the field K has at least n elements. Then there exists a point $P \in E$, such that the orbit $G(P) = \{g(P) \mid g \in G\}$ of P contain n elements. (**Hint:** For P choose an arbitrary element in E, which is not in the union of the fix point sets of $g \in G$, $g \neq \mathrm{id}$. The union of at most $n-1$ proper affine subspace of E cannot be the whole E. See exercise 2.2.)

12.10. Let E be an affine space over the K-vector space V.

a). Let $U \subseteq V$ be a K-subspace of V. The set E/U of all affine subspaces $U+P$, $P \in E$, of E which are parallel to U is an affine space over the quotient space V/U with respect to the operation

$$(x+U) + (U+P) = U + (x+P)\,,$$

$x+U \in V/U$, $U+P \in E/U$. The natural projection $E \to E/U$ is an affine map and its linear part is the natural projection of $V \to V/U$.

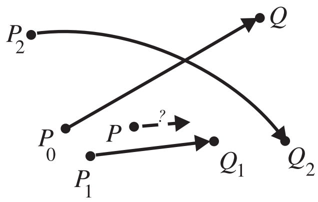
b). Let W be another K-vector space. The set of all affine maps $E \to W$ form a K-subspace of the vector space W^E and is isomorphic to th vector space $W \times \mathrm{Hom}_K(V, W)$. (**Hint:** $f \mapsto \big(f(O)\,,\, f_0\big)$ with a fixed point $O \in E$ is an isomorphism.)

c). Let F be affine space over the K-vector space W. The set of all affine maps $f : E \to F$ form an affine space over the K-vector space of the affine maps $g : E \to W$ (see the part b) above) with the operation $(g, f) \longmapsto g + f$, where the sum $g+f$ is defined by $(g+f)(P) := g(P) + f(P)$.

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

Test-Exercises

T12.1. a). Construct [1]) the image point $f(P)$ of the point P under the affinity f, which maps the affine basis P_0, P_1, P_2 onto the points Q_0, Q_1, Q_2.



b). Injective affine maps preserves ratios.

c). The bijective map $(z_1, z_2) \longmapsto (\overline{z}_1, \overline{z}_2)$ of $\mathbb{A}^2(\mathbb{C}) = \mathbb{C}^2$ onto itself is a collineation, which is not an affine collineation.

d). The bijective map $(z_1, z_2) \longmapsto (z_1, \overline{z}_2)$ of $\mathrm{A}^2(\mathbb{C})$ onto itself is not a collineation.

e). Let E be an affine space over the $\mathbb{C}$-Vektorraum V of dimension ≥ 1 and $h : V \to V$ be a bijective $\mathbb{C}$-anti-linear map from V onto itself. If P_0, Q_0 are points in E, then the map f with $f(P) := h\big(\overrightarrow{P_0P}\big) + Q_0$, $P \in E$, is a collineation of E, which is not affine. (**Remark:** The part c) is a special case of this construction.)

T12.2. (Description of the affine group) Let V be a K-vector space and consider V as affine space over itself. The affine maps of V into self are exactly the maps of the form $x \mapsto v + h(x)$, $x \in V$, where h is an endomorphism of V and $v \in V$ is a fixed vector (corresponding to the affine map). This affine map is usually denoted by (v, h). Then the composition of (v, h) and (u, g) is $(v, h) \circ (u, g) = (v + h(u), hg)$. The linear map corresponding to the affine map (v, h) is h. Therefore (v, h) is an affinity if and only if $h \in \mathrm{GL}(V)$; and in this case the inverse map of (v, h) is $(v, h)^{-1} = (-h^{-1}(v), h^{-1})$. Therefore the affine group $\mathrm{A}(V)$ of V (in particular, the affine group $\mathrm{A}_n(K)$ of $K^n = \mathbb{A}^n(K)$) can be described by using the additive group of V and the automorphism group $\mathrm{GL}(V)$ of the K-vector space V. The affine group $\mathrm{A}(V)$ is the semi-direct product $V \rtimes \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ operates on V in a natural way. This group is the subgroup of the permutation group $\mathfrak{S}(V)$ of V, generated by the translations and the K-linear automorphisms of V. See T11.????.

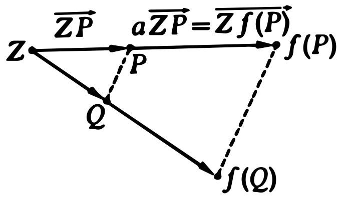
The affine group $\mathrm{A}(V)$ *has a natural embedding in the automorphism group* $\mathrm{GL}(V \oplus K)$. The affinity $(v, h) \in \mathrm{A}(V)$, $v \in V$, $h \in \mathrm{GL}(V)$, will corresponds to the automorphism $\binom{x}{a} \longmapsto \binom{hx+av}{a}$, $x \in V$, $a \in K$ of $V \oplus K$. This (in analogy with matrix notation) will be denoted by $\begin{pmatrix} h & v \\ 0 & 1 \end{pmatrix}$. (**Remark:** For an arbitrary A-module V over an arbitrary ring A, the semi-direct product $V \rtimes \mathrm{Aut}_A(V)$ is called the affine group $\mathrm{A}(V) = \mathrm{A}_A(V)$ of V and is identified with a subgroup of $\mathfrak{S}(V)$. In particular, we put $\mathrm{A}_n(A) = \mathrm{A}(A^n) = \mathrm{A}\big(\mathbb{A}^n(A)\big)$ for every $n \in \mathbb{N}$ and every ring A.)

T12.3. (Magnifications and point-reflections) Let E be an affine spaceover the K-vector space. An affinity f of E is called a (centric) magnification or a homothecy of E if the linear part f_0 is a homothecy of V. If $a \in K^\times$ is the *magnification ratio* of f_0, i.e. if $f_0 = a\,\mathrm{id}_V$, then a is called the magnification ratio of f.

a). The magnifications of E with magnification ratio 1 are the translations of E.

b). If the magnification ratio a of the magnification f is different from 1, then f has exactly one fixed point Z. This fixed point is $Z = (1-a)^{-1}\overrightarrow{OO'} + O$ and is called the center of the magnification f. (**Hint:** Let $O \in E$ and let $O' := f(O)$. For an arbitrary point $P \in E$, we have $f(P) = a\,\overrightarrow{OP} + O' = a\,\overrightarrow{OP} + \overrightarrow{PO'} + P$. Therefore P is a fixed point of f if and only if $0 = a\,\overrightarrow{OP} + \overrightarrow{PO'} = (a-1)\overrightarrow{OP} + \overrightarrow{OO'}$ and so $\overrightarrow{OP} = (1-a)^{-1}\overrightarrow{OO'}$. This fixed point is equal to $Z = (1-a)^{-1}\overrightarrow{OO'} + O$.

[1]) For this construction one is allowed to use only line-segments and parallel lines through the constructed points.



)

c). A centric magnification of E with magnification factor -1 is called a p o i n t - r e f l e c t i o n of E (if $-1\neq 1$ in K, i.e. Char$K\neq 2$).

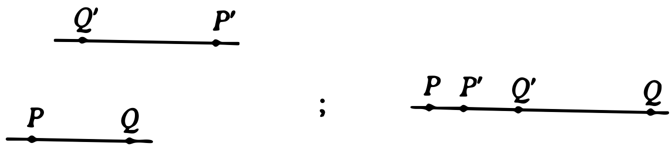
d). The centric magnifications of the affine space E form a subgroup (in fact a normal subgroup) of the affine group $\mathrm{A}(E)$ of E.

e). Product of two magnifications with magnification ratio a and b respectively, is a magnification with magnification ratio ab. Moreover, if the centers of the given magnifications are distinct then the center of the product lies on the line passing through these centers.

f). The magnifications with a fixed magnification factor along with the identity form a subgroup of all magnifications and this subgroup is isomorphic to the multiplicative group $K^\times$ of K.

g). An affinity f is a magnification if and only if the image of a line is a line parallel to it. (**Hint:** consider the tow cases whether or not f has a fixed point.)

h). Let P, Q, P', Q' be points in E with $P\neq Q$ and $P'\neq Q'$. If the line-segments PQ and $P'Q'$ are parallel, then there exists a unique magnification with $P\mapsto P'$ and $Q\mapsto Q'$. Construct the center of this magnification in the following cases examples:



i). Suppose that $1+1\neq 0$ in K, i.e. Char $K\neq 2$. Then the product of two point- reflections of an K-affine space is a translation. Conversely, every translation of an K-affine space is a product of two point reflection.

T12.4. Let E and F be K-affine spaces. The set of all those points in E at which the two affine maps from E into F are equal is an affine subspace of E. For example:

(1) The set of all fixed points of a affine maps from E into itself is an affine subspace of E.
(2) If two affine maps from E in F are equal on an affine generating system of E, then they are equal.

T12.5. Let E be an affine space over the K-vector space V.

a). If f is an affinity of E and if $t=t_x$ is the translation of E by the vector $x\in V$, then $f\circ t_x\circ f^{-1}=t_{f_0(x)}$. In particular, f commute with the translation t_x if and only if x is a fixed point of the linear part f_0 of f.

b). Let x_i, $i\in I$, be a generating system of V. An affinity of E which commute with all the translations defined by the vectors x_i, $i\in I$, is itself a translation.

T12.6. Let E be a finite dimensional affine space over the K-vector space V and let h be a K-endomorphism of V. Then the following statements are equivalent:

(1) Every affine map from E into itself with linear part h has exactly one fixed point.
(2) Every affine map from E into itself with linear part h has at least one fixed point.
(3) Every affine map from E into itself with linear part h has at most one fixed point.
(3′) There exists an affine map from E into itself with linear part h and exactly one fixed point.
(4) $h-\mathrm{id}_V$ is an automorphism of V.

(**Remark:** (In the infinite dimensional case (1) and (4) are still equivalent. The condition (4) will be used to formulate : 1 *is not a Spectral value of* f.)

T12.7. Let E be an affine space over the K-vector space V and let $f: E\to E$ be an afine map. Then f is an affine involution (resp. pseudo-reflection) (see 12.5 and 12.6) if and only if f has a fixed point and the linear part f_0 of f is an affine involution (resp. pseudo-reflection). (**Hint:** Let $O \in E$ be a fixed point of f. Then the diagramm

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & F \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ V & \xrightarrow{\ f_0\ } & W \end{array}$$

is commutative, where $g : V \to E$ is the affine isomorphism defined by $x \mapsto x + O$.)

T12.8. Every affine subspace of the codimension m of K^n is the solution space of a linear system over K with consisting of m equations and n unknowns (this will necessarily have rank m), vgl. 5.E, Aufg 13.

a). Give a system of linear equations over $\mathbb{R}$ with the solution space is the affine subspace $U + (1, 1, 1, 1) \subseteq \mathbb{R}^4$, where U is generated by the vectors $(1, 0, 1, 1)$ and $(0, -1, 2, 0)$.

b). Give a system of linear equations over $\mathbb{C}$ with the solution space is the affine hull of the points $(1, 1 - \mathrm{i}, 0)$, $(-\mathrm{i}, 0, 2 - \mathrm{i})$ and $(0, -\mathrm{i}, 1)$ im $\mathbb{C}^3$.

T12.9. Let $f : E \to F$ be an affine map, P_i, $i \in I$ be a family of points in E and $(a_i) \in K^{(I)}$ be a family of weights with $\sum a_i \neq 0$. Then the image of the center of mass of the points P_i, $i \in I$ with weights a_i, $i \in I$ under f is the center of mass of the points $f(P_i)$, $i \in I$ with weights a_i, $i \in I$.

T12.10. Let $g : E \to E$ be an affinity of an n-dimensional affine space over a K-vector space V and let $F := \operatorname{Fix} g := \{P \in E \mid g(P) = P\}$ be the fixed point set of g. This is an affine subspace of E (see ????).

a). If $F := \operatorname{Fix} g = \emptyset$ i.e. g has no fixed point, then there exists a shearing $s : E \to E$ of E such that sg has a fixed point, i.e. $\operatorname{Fix}(sg) \neq \emptyset$. (**Proof:** Let $P_0 \in E$ and let $P_1 := g(P_0)$. Then $P - 1 \neq P_0$ (since g has no fixed point), i.e. $v_1 := \overrightarrow{P_0 P_1} \neq 0$. Extend v_1 to a basis $v_1, \ldots, v_n$ of V. Then $P_0, P_1, P_i := P_0 + v_i$, $i = 2, \ldots, n$ form an affine basis of E. Put $P_0' := P_2 + v_1$. Then $P_0', P_1, \ldots, P_n$ form an affine basis of E (since the four points P_0, P_2, P_0', P_1 form a paralleogramm; $\overrightarrow{P_0' P_2} = -v_1$ and $\overrightarrow{P_0' P_1} = \overrightarrow{P_0' P_2} + \overrightarrow{P_2 P_1} = -\overrightarrow{P_0 P_1} + \overrightarrow{P_2 P_1} = \overrightarrow{P_0 P_2} = -v_2$.) Now, define $s : E \to E$ by $s(P_0') = P_0'$, $s(P_1) = P_0$ and $s(P_i) = P_i$ for $i = 2, \ldots, n$. The fixed point set of s is the hyperplane H_s spanned by the points $P_0', P_2, \ldots, P_n$ and the vector $\overrightarrow{P_1 s(P_1)} = \overrightarrow{P_1 P_0'} = -v_1 = \overrightarrow{P_0' P_2}$ is parallel to H_s, i.e. belongs to the vector space corresponding to H_s. Therefore s is a shearing and by construction, the point $P_0 \in \operatorname{Fix}(sg)$.)

b). Suppose that $\operatorname{Dim}(\operatorname{Fix} g) = k \in \{0, 1, \ldots, n-2\}$ and $P_0, \ldots, P_k \in F$ is an affine basis of F and $P_{k+1} \notin F$. Let F' be the affine subspace of E generated by $P_0, \ldots, P_k, P_{k+1}$. Then

(1) $\operatorname{Dim} F' = k + 1$, $g(P_{k+1}) \notin F$ and $P_0, \ldots, P_k, g(P_{k+1})$ are affinely independent.

(2) if $g(P_{k+1}) \notin F'$, then there exists a shearing $s : E \to E$ of E such that $F' \subseteq \operatorname{Fix}(sg)$.

(**Proof:** Since $g(P_{k+1}) \notin F'$, $P_0, \ldots, P_{k+1}, P_{k+2} := g(P_{k+1})$ are affinely independent and so extend to an affine basis $P_0, \ldots, P_{k+2}, P_{k+3}, \ldots, P_n$ E. Define $P_{k+1}' := P_0 + \overrightarrow{P_{k+1} P_{k+2}}$ (note $k \leq n-2$). Then, the points $P_0, \ldots P_k, P_{k+1}', P_{k+2}, \ldots, P_n$ form an affine basis of E (since the points $P_0, P_{k+1}', P_{k+2}, P_{k+1}$ form a parallelogram and by definition, we have $\overrightarrow{P_0 P_{k+1}'} = \overrightarrow{P_{k+1} P_{k+2}}$, and $\overrightarrow{P_{k+1}' P_{k+2}} = \overrightarrow{P_0 P_{k+1}}$.) Now, define $s : E \to E$ by $s(P_{k+2}) = P_{k+1}$, $s(P_{k+1}') = P_{k+1}'$ and $s(P_i) = P_i$ for $i = 0, \ldots, k, k+3, \ldots, n$. The fixed point set of s is the hyperplane H_s spanned by the points $P_0, \ldots, P_k, P_{k+1}', P_{k+3}, \ldots, P_n$ and the vector $\overrightarrow{P_{k+2} s(P_{k+2})} = -\overrightarrow{P_{k+1}' P_0}$ is parallel to H_s, i.e. belongs to the vector space corresponding to H_s. Therefore s is a shearing and by construction, the points $P_0, \ldots, P_k, P_{k+1} \in \operatorname{Fix}(sg)$.)

(3) if $g(P_{k+1} \in F'$, then there exist shearings $s : E \to E$ and $s' : E \to E$ of E such that $F' \subseteq \operatorname{Fix}(s'sg)$.

(**Proof:** Extend $P_0, \ldots, P_k, g(P_{k+1})$ to an affine basis $P_0, \ldots, P_k, g(P_{k+1}), P_{k+2}, \ldots, P_n$ of E and define $Q := g(P_{k+1}) + \overrightarrow{P_0 P_{k+2}}$. Then, since $\overrightarrow{P_0 P_{k+1}} \notin \sum_{i=1}^{k} K \overrightarrow{P_0 P_i}$, we have $P_0, \ldots, P_k, Q, P_{k+2}, \ldots, P_n$ is an affine basis of E. Now, define $s_1 : E \to E$ by $s_1(P_i) = P_i$ for $i = 0, \ldots, k, k+2, \ldots, n$ and $s_1(g(P_{k+1})) = Q$. Then the fixed set $\operatorname{Fix} s_1$ of s_1 is the hyperplane H_s generated by $P_0, \ldots, P_k, P_{k+2}, \ldots, P_n$ and the vector $\overrightarrow{g(P_{k+1}) Q} = \overrightarrow{P_0 P_{k+2}}$ is parallel to H_s, i.e. belongs to the vector space corresponding to H_s. Therefore s_1 is a shearing and by construction, the points $P_0, \ldots, P_k, P_{k+1} \in \operatorname{Fix}(s_1 g)$.

Now, construct a second shearing $s_2 : E \to E$ as in the part (2) by replacing $g(P_{k+1})$ by Q. Since $s_1(g(P_{k+1})) = Q$ and $s_2(Q) = P_{k+1})$ and both fix $P_0, \ldots, P_k$, all the points $P_0, \ldots, P_k$ and P_{k+1} are fixed by $s_2 s_1 g$.)

T12.11. Let K be a finite field with q elements.

a). The automorphism group of an n-dimensional K-vector space V has the order

$$(q^n-1)(q^n-q)\cdots(q^n-q^{n-1}) = q^{\frac{1}{2}n(n-1)}\,(q^n-1)(q^{n-1}-1)\cdots(q-1)\,.$$

(**Hint:** The number of automorphisms of V is equal to the number of bases $(v_1,\ldots,v_n)$ of V. To compuet this numebr use (for example) the following combinatorics principle.)

Deduce that : The number of m-dimensional K-subspaces of V is

$$\frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}\,.$$

(**Remark:** This number is the value of the *Gauss polynomial* $G_m^{[n]}$ at the place q.)

b). The affine group of an n-dimensional K-affine space has the order

$$q^{\frac{1}{2}n(n+1)}\,(q^n-1)(q^{n-1}-1)\cdots(q-1)\,.$$

(**Remark:** We shall illustrate this with a simple example. Let $K=\mathbf{K}_2$ be the field with two elements and $n=2$. The group $\mathrm{GL}(\mathbf{K}_2^2)$ has the order $(4-1)(4-2)=6$ and it isomorphic to a subgroup of $\mathfrak{S}\big(\mathbf{K}_2^2-\{0\}\big)$ [2]) Therefore $\mathrm{GL}(\mathbf{K}_2^2)\cong\mathfrak{S}_3$. the affine group $\mathrm{A}(\mathbf{K}_2^2)$ has the order $4\cdot 6=24$ and isomorphic to the subgroup $\mathrm{A}(\mathbf{K}_2^2)\cong\mathfrak{S}_4$ of $\mathfrak{S}(\mathbf{K}_2^2)$. Further, $\mathrm{A}(\mathbf{K}_2^2)$ contain the additive group of $\mathbf{K}_2^2$ as a normal subgroup and the $\mathrm{GL}(\mathbf{K}_2^2)$ is the factor group. Prove that this in the only normal subgroup of order 4 in $\mathrm{A}(\mathbf{K}_2^2)$. (The image of such a normal subgroup in $\mathrm{GL}(\mathbf{K}_2^2)\cong\mathfrak{S}_3$ is necessarily trivial.) Therefore : *The group* $\mathfrak{S}_4$ *contain exactly one normal subgroup* $\mathfrak{V}_4$ *of order* 4. *Further,* $\mathfrak{V}_4\cong\mathbb{Z}/\mathbb{Z}2\times\mathbb{Z}/\mathbb{Z}2$ *and* $\mathfrak{S}_4/\mathfrak{V}_4\cong\mathfrak{S}_3$. The group $\mathfrak{V}_4\subseteq\mathfrak{S}_4$ is called the Klein's 4-group. This group contain (other than identity) the permutations $\binom{1\,2\,3\,4}{2\,1\,4\,3}$, $\binom{1\,2\,3\,4}{3\,4\,1\,2}$, $\binom{1\,2\,3\,4}{4\,3\,2\,1}$.)

T12.12. Let K be either the field $\mathbf{K}_0:=\mathbb{Q}$ or the finite field $\mathbf{K}_p:=\mathbb{Z}/\mathbb{Z}p$, p prime number. Then for every K-vector space V the group $\mathrm{Aut}_K V$ of all is equal the group of all automorphisms of the additive group of V. The affine group $\mathrm{A}_K(V)$ of V is the full-holomorph $\mathrm{Hol}\,(V)$ of the additive group of V. [3]) (**Remark:** Similarly, the affine group of an arbitrary abelian group G, considered as $\mathbb{Z}$-module, is the full-holomorph $\mathrm{Hol}\,(G)$ of G, see. Beispiel 6.E.9. As an example we consider the full-holomorph of the cyclic group $\mathbb{Z}/\mathbb{Z}p$ of order p, which is the group $\mathrm{A}_1(\mathbf{K}_p)=\mathbf{K}_p\rtimes\mathbf{K}_p^\times$, where the multiplicative group $\mathbf{K}_p^\times$ acts by multiplication on (the additive group) $\mathbf{K}_p=\mathbb{Z}/\mathbb{Z}p$. Since $\mathbf{K}_p^\times$ is cyclic and so for every divisor t of $|\mathbf{K}_p^\times|=p-1$ has a unique (cyclic) subgroup U_{pt} of order t. The subgroup

$$\mathbf{F}_{pt}:=\mathbf{K}_p\rtimes U_{pt}\subseteq\mathbf{K}_p\rtimes\mathbf{K}_p^\times=\mathrm{A}_1(\mathbf{K}_p)$$

and every group which is isomorphic to this group is called a Frobenius-group. A Frobenius-group is uniquely determined [4]) upto isomorphism by its order pt.

A group G is isomorphic to a Frobenius-group if and only if it has a nomal subgroup N of prime order and the centraliser of N in G is N itself. (proof! [5]))

---

[2]) $f\mapsto f|\big(\mathbf{K}_2^2-\{0\}\big)$ is a canonical embedding of $\mathrm{GL}(\mathbf{K}_2^2)$ in $\mathfrak{S}\big(\mathbf{K}_2^2-\{0\}\big)$.

[3]) The additive groups of the vector spaces over $\mathbb{Z}/\mathbb{Z}p$, p prime, are precisely the elementary abelian p-groups, and these are abelian groups H with $\mathrm{Ord}\,x=p$ for all non-zero elements $x\in H$, i.e. with $px=0$ for all $x\in H$.

[4]) The Frobenius group $\mathbf{F}_{pt}$ is clearly determined by the prime p which is the biggest prime factor of its order and has the normal subgroup $\mathbf{K}_p$ of $\mathbf{F}_{pt}$ and by the divisor t of $p-1$. As the cyclic group $\mathbf{K}_p^\times$ has exactly one subgroup of order t. Its action via multiplication on $\mathbf{K}_p$ determines therefore the resulting semidirect product.

[5]) Suppose that G is a group with a normal subgroup N of prime order (we may assume that $(\mathbf{K}_p,+)$ for some p) and such that the centralizer of N in G is N. Then, by definition of the centralizer of a normal subgroup, N is the kernel of $\kappa: G\to\mathrm{Aut}\;N$ the conjugate action of G on N. Therefore G/N is isomorphic to the subgroup $\mathrm{im}\;\kappa=H'$ of the group of automorphisms $\mathrm{Aut}\;N(\cong\mathbf{K}_p^\times)$ of N. Now, since $\mathbf{K}_p^\times$ is a cyclic group of order $p-1$, the subgroup H' is also cyclic of order t (a divisor of $p-1$) generated by

Now, let G be an arbitrary group of order pq, where $q < p$ are prime numbers. Then G has a subgroup N of order p and a subgroup H of order q. The group N is uniquely determined (otherwise G has at least p^2 elements) and in particular, N is normal in G. Further, since $N \cap H = \{e\}$, the group G is a semi-direct product of N and H. For the operation (by conjugation) of H on N, the follwoing two casses are possible:

(1) The operation is trivial. Then G is commutative and so cyclic of the order pq.

(2) The operation is not trivial. Then it must be faithful, since H is cyclic of prime order and so $q = |H|$ is a divisor of $|\mathrm{Aut}\, N| = p - 1$ and then G is isomorphic to the Frobenius-group $\mathbf{F}_{pq}$.

We have proved that : *A group of order pq, p, q prime numbers with $q < p$ is either cyclic or (in the case q is a divisor of $p-1$) isomorphic to a Frobenius-group* $\mathbf{F}_{pq}$. – For example, a group of order 15 is cyclic and a group of order 21 is either cyclic or isomorphic to a Frobenius-group $\mathbf{F}_{21}$.)

T12.13. (Classical space-time-world) Perhaps the greatest obstacle to understand the theories of *special* and *general relativity* [6]) arises from the difficulty in realising that a number of previosuly held basic assumptions about the nature of space and time are wrong. We therefore spell-out some key assumptions about space and time. We can consider space and time ($\equiv$ space-time [7])) to be a continuum composed of events, where each event can be thought as a point of space at an instant of time.

Up to now we have only considered the *universe* S over the vector space V_S of translations, and time was ignored. Classically, time is a real affine line T. The corresponding vector space is denoted by V_T; for the measurement of time, we choose a basis τ of V_T, pointing into the "future", i.e. for given moments t_1 and t_2 in T, we say that "t_1 comes before t_2" if the vector $\overrightarrow{t_1t_2}$ has a representation $a\tau$ with a positive real number a (arrow in the direction of time). The motion of a free particle on a line in the universe gives an isomorphism of this line onto T. The most naive description of the space-time-world as a whole is done through the four-dimensional product space $S \times T$ which is, in a natural way, an affine space over the $\mathbb{R}$-vector space $V_S \times V_T$. Both the projections of $S \times T$ onto S and T are affine maps. They associate to every world-point in $S \times T$ its position resp. its time. The fibres of these projections are the points with the same position resp. time.

It has been known from early times – at least from the time of Aristotle – that it does not make sense to talk about two events taking place at different times at the same place. Description of position is only possible relative to a frame of reference; one cannot distinguish any one of these frames of reference as a fixed frame of reference. On the other hand, in the area of classical physics one has the concept of simultaneousness: Two distinct world-points are not simultaneous if and only if (at least in the mental experiment) the same mass-point can occupy both these world-points.

Therefore one describes the classical space-time-world as a four dimensional real affine space E with an affine (non-constant) map $z : E \to T$ from E onto the time T. For an event $P \in E$, we call $z(P)$ the time at which the event P takes place. The fibres of the affine map z define the space-directions. Our universe, which we have handled so far, was always such a fibre. All these fibres are parallel to the three-dimensional subspace V_S of the vector space V_E corresponding to E.

Two world-points P and Q in E differ from each other by the vector $\overrightarrow{PQ}$. P and Q are simultaneous if and only if $\overrightarrow{PQ} \in V_S$. Therefore the vectors in V_S are called space-like vectors. Every vector in V_E,

---

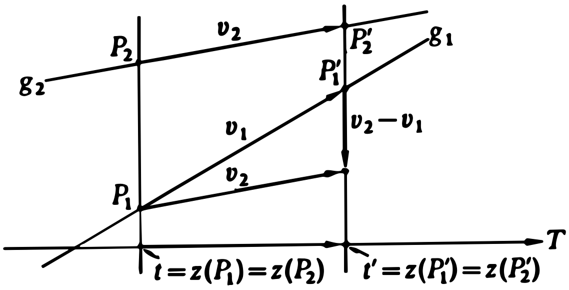
$a' \in H'$. Let $a \in \kappa^{-1}(a') \subseteq G$ be a preimage of a'. Then the residue class of a^t in G/N is zero, i.e. a^t is contained in N. Therefore $(a^p)^t = a^{pt} = (a^t)^p = e_G$ and so the order of a^p in G is a divisor of t and $\kappa(a^p) = \kappa(a)^p = (a')^p$ As p and $p-1$ have no common divisor, a' and $(a')^p$ have the same order in $\mathbf{K}_p^\times$ and so $(a')^p$ also generate H'. Therefore the subgroup H of G generated by a^p has order t and is canonically isomorphic to $H' \cong G/N \subseteq \mathbf{K}_p^\times$ and hence by definition of a semidirect product, this yields $G \cong \mathbf{F}_{pt}$.

[6]) The general theory of relativity is one of the greatest intellectual achievements of all time. Its originality and unorthodox approach exceed that of special relativity. And for so more than special relativity, it was almost completely the work of a single man, ALBERT EINSTEIN (1879-1955). The philosophic impact of relativity theory on the thinking of man has been profound and the vistas of science opened by it are literally endless.

[7]) HERMANN MINKOWSKI (1864-1909) referred to space-time as the world, hence events are world-points and a collection of events giving history of a particle is a world-line. Physical laws on the interaction of particles can be thought of as the geometric relation between the world-lines. In this sense Minkowski maty be said to have *geometrized* physics.

which is not a space-like vector, is called *time-like*. The world-points representing the motion of a free particle m_1 (which is not subject to any outer forces), form an affine line $g_1 = \mathbb{R}v_1 + P_1$ in E, the so called *world-line* of these mass-points. It is parallel to the line $\mathbb{R}v_1$ in V_E generated by some time-like vector v_1 (*Galilean law of inertia*). Then the line g_1 representing the time and the affine subspace $V_S + P_1$ give a decomposition of E into space and time (as above). After normalising the vector v_1 by the condition $z_0(v_1) = \tau$, where z_0 is the linear part of z, this vector v_1 is called the *absolute* or *four-velocity* of the mass-point under consideration.

If m_2 is another mass-point with the absolute velocity v_2 (moving freely without being subject to outer forces), then $v_2 - v_1 \in V_S$ is a space-like vector. It is called the *relative velocity of m_2 with respect to m_1*.



The simultanousness as defined above requires arbitrary large relative velocities. Since observations suggest that arbitrary large velocities cannot occur, one tries to abandon the notion of simultanousness. A first step in this direction is the special theory of relativity.

As automorphisms of the classical space-time-world E described above we shall consider the affinities f of E, which are compatible with the time map $z : E \to T$. By this we mean that there exists an affinity $f_T : T \to T$ (which is necessarily uniquely determined) such that $z \circ f = f_T \circ z$:

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle z} \\ T & \xrightarrow{\;f_T\;} & T \end{array}$$

These automorphisms f of E form a subgroup G of the affine group $\mathrm{A}(E)$ of E. This subgroup G is called the *affine Galilean group*. An affinity f in $\mathrm{A}(E)$ belongs to G if and only if its linear part maps the vector space V_S of the spacelike vectors into itself. By G_0 we denote the subgroup of automorphisms h of V_E with $h(V_S) \subseteq V_S$. Then the map $G \to G_0$ defined by $f \mapsto f_0$ is a surjective group homomorphism, and its kernel is the group $\mathrm{T}(E)$ of all translations of E. In particular, $G/\mathrm{T}(E) \cong G_0$.

Sometimes the subgroup of all $f \in G$ such that the time-part f_T is the identity, is also called the affine Galilean group.

T12.14. (*Collineations*) Let E and F be affine spaces (not necessarily over the same field). A bijective map $f : E \to F$ is called an *isomorphism* of E onto F if the map $E' \mapsto f(E')$ is a bijective map from the set of all affine subspaces of E onto the set of all affine subspaces of F. An isomorphism of E onto itself is called a *collineation* of E. The group of all collineations of E is denoted by $\mathrm{K}(E)$. (If $\mathrm{Dim}\, E = 1$, then $\mathrm{K}(E) = \mathfrak{S}(E)$.)

Let $f : E \to F$ be a bijective map of the K–affine space E onto the L–affine space F. Then

a). If f is an isomorphism and $P_i,\ i \in I$ is an affine basis of the affine subspace E' of E, then $f(P_i),\ i \in I$ is an affine basis of $f(E')$. In particular, $\mathrm{Dim}\, f(E') = \mathrm{Dim}\, E'$ for every affine subspace E' of E. (**Proof:** Let F' be the affine subspace of F gereated by $f(P_i),\ i \in I$. Then $F' \subseteq f(E')$. Since $P_i \in f^{-1}(F')$ for all $i \in I$, $E' \subseteq f^{-1}(F')$ or $f(E') \subseteq F'$. Therefore $f(E') = F'$ and $f(P_i),\ i \in I$, is an affine generating system for $f(E')$. Suppose that $f(P_i),\ i \in I'$, with $I' \subseteq I$ is an affine basis of $f(E')$. Then by the above proof $P_i,\ i \in I'$ is an affine generating system of $f^{-1}(f(E')) = E'$ and hence $I' = I$. Therefore the family $f(P_i),\ i \in I$ is also affinely independent and so an affine basis of $f(E')$.)

b). If $|K| > 2$, then f is an isomorphism if and only if the f–image of every affine line in E is an affine line in F. (**Proof:** In view of a) above and 43.3, it is enough to prove that: f induces a bijective map between the set of all lines in E resp. in F, if the f–image of every line in E is a line in F. Further, it is enough to prove that every line g' in F is the image of a line in E. For this, let $P',\ Q'$ be two distinct points on g' and $P,\ Q$ be the preimages of $P',\ Q'$ in E. Let g be the line generated by P and Q in E. Then $f(g)$ is a line in F with $P',\ Q' \in f(g)$ and hence $f(g) = g'$.)

c). If $|K| = 2$, then f is an isomorphism if and only if the f–image of every affine plane in E is an affine palne in F. (**Proof :** Similar to that of the part b) above with the following – **Remark :** A subset of a $\mathbf{K}_2$-affine space is an affine subspace if and only if for every theree affinely independent points, the fourth one is also contained in the affine plane generated by them.)

T12.15. (Fundamental theorem of affine geometry) *Let $f : E \to F$ be a bijective map from the K–affine space E over V onto the L–affine space F over W. Suppose that* $\operatorname{Dim} E \geq 2$. *Then f is an isomorphism if and only if there exist an isomorphism $\varphi : K \to L$ of fields and a bijective φ–linear map $\overline{f} : V \to W$ such that the canonical diagramm*

$$\begin{array}{ccc} V \times E & \longrightarrow & E \\ {\scriptstyle \overline{f}\times f}\downarrow & & \downarrow{\scriptstyle f} \\ W \times F & \longrightarrow & F \end{array}$$

is commutative.

(**Proof :** If there exist φ and $\overline{f}$ as given in the theorem, then f is an isomorphism, since $\overline{f}$ induces a bijective map of the set of all K–subspaces of V onto the set of all L–subspaces of W.

Conversely, suppose that f is an isomorphism. Let $O \in E$ be a fixed point. It is enough to show that the map $\overline{f} : V \to W$ defined by $x \mapsto \overrightarrow{f(O)f(x+O)}$ is a bijective and φ–linear for a suitable isomorphism $\varphi : K \to L$ of fields. The diagramm

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & F \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ V & \xrightarrow{\ \overline{f}\ } & W \end{array}$$

is commutative, where g_1 resp. g_2 are affine isomorphisms defined by $x \mapsto x + O$ resp. $y \mapsto y + f(O)$. Therefore $\overline{f}$ is also an isomorphism of the affine spaces V and W. We may therefore assume that $f = \overline{f}$.

Then $f(0) = 0$. Further, f maps parallel lines onto parallel lines, since two distinct lines are parallel if and only if they lie in a plane and donot intersect, vgl. Aufgabe 7. Therefore it follows that $f(Kx + y) = Lf(x) + f(y)$ for arbitrary $x,\ y \in V$; For, if $x \neq 0$, then the lines $f(Kx) = Lf(x)$ and $f(Kx + y)$ are parallel and since $f(y) \in f(Kx + y)$, we necessarily have $f(Kx + y) = Lf(x) + f(y)$. Finally, we remark that the image under f of any two linearly independent vectors $x,\ y \in V$ are linearly independent in W, since $f(y)$ does not belong to $f(Kx) = Lf(x)$, see also 43.8(1).

(1) *f is additive* : Let $x, y \in V$. If either $x = 0$ or $y = 0$, then clearly $f(x + y) = f(x) + f(y)$.

(1.a) Case : *$x,\ y$ are linearly independent*: Then $x + y$ (resp. $f(x) + f(y)$) is the point of intersections of the lines $Kx + y$ and $Ky + x$. (resp. $Lf(x) + f(y)$ and $Lf(y) + f(x)$) and so $\{f(x) + f(y)\} = (Lf(x) + f(y)) \cap (Lf(y) + f(x)) = f(Kx + y) \cap f(Ky + x) = f((Kx + y) \cap (Ky + x)) = \{f(x + y)\}$ as required.

(1.b) Case : *$x,\ y$ are linearly dependent*: Since both $x,\ y$ are non-zero, there exists (since $\operatorname{Dim}_K V \geq 2$) a vector $z \in V$ such that $x,\ z$ are linearly independent. Then $x,\ y + z$ and $y,\ z$ are also linearly independent. If $x + y \neq 0$, then $x + y,\ z$ are linearly independent and so (by the earlier proof) $f(x + y) + f(z) = f((x+y)+z) = f(x+(y+z)) = f(x)+f(y+z) = f(x)+f(y)+f(z)$, i.e. $f(x+y) = f(x)+f(y)$. If $x + y = 0$, then (by the earlier proof) $f(z) = f(x + (y + z)) = f(x) + f(y + z) = f(x) + f(y) + f(z)$, i.e. $f(x + y) = f(x) + f(y) = 0$.

(2) *Definition of φ*: Let $x \in V,\ x \neq 0$ be fixed and define $\varphi : L \to K$ by the equation $f(ax) = \varphi(a)f(x)$. Since $f(Kx) = Lf(x)$, it follows that φ bijective.

(3) *f is φ-semi-linear* : i.e. *$f(ay) = \varphi(a)f(y)$ for all $a \in K$ and all $y \in V$.* This is clear if $a = 0$ or $y = 0$.

(3.a) Case : *$x,\ y$ are linearly independent* : Then ay (resp. $\varphi(a)f(y)$) is the point of intersection of the lines Ky and $K(y - x) + ax$ (resp. $Lf(y)$ and $L(f(y) - f(x)) + \varphi(a)f(x)$). Therefore $\{\varphi(a)f(y)\} = Lf(y) \cap (L(f(y - x)) + \varphi(a)f(x)) = f(Ky \cap (K(y - x) + ax)) = \{f(ay)\}$.

(3.b) Case : $x,\ y$ *are linearly dependent* : Let $z \in V$ be as in (1.b). Then by the additivity of f and the earlier case (3.a), we have $f(ay) = f(a(y+z)-az) = f(a(y+z)) - f(az) = \varphi(a)f(y+z) - \varphi(a)f(z) = \varphi(a)f(y) + \varphi(a)f(z) - \varphi(a)f(z) = \varphi(a)f(y)$.

$f(az) = \varphi(a)f(z)$ and $f(a(y+z)) = \varphi(a)f(y+z) = \varphi(a)f(y) + \varphi(a)f(z)$.

(4) φ *is an isomorphism of fields*: Let $a, b \in K$ and let $x \in V,\ x \neq$. Then by the definition of φ and the addivity of f, we have: $\varphi(a+b)f(x) = f((a+b)x) = f(ax) + f(bx) = \varphi(a)f(x) + \varphi(b)f(x) = (\varphi(a) + \varphi(b))\, f(x)$ and so $\varphi(a+b) = \varphi(a) + \varphi(b)$. Similarly, we have : $\varphi(ab)f(x) = f((ab)x) = \varphi(a)f(bx) = \varphi(a)\varphi(b)f(x)$ and so $\varphi(ab) = \varphi(a)\varphi(b)$. Finally, from $f(x) = f(1\cdot x) = \varphi(1)f(x)$, we have $\varphi(1) = 1$. This proves that φ is an isomorphism of fields.)

More generally, a bijective map of E onto itself is called a collineation of E if the image of every affine line in E under f is again an affine line in E. If E itself is a line, then naturally every bijective map of E onto itself is a collineation.

If Dim $E \geq 2$, then the fundamental theorem of affine geometry gives a closed relation between collineations of E and affine maps of E. The fields $\mathbb{Q}$, $\mathbb{R}$ and Z_p p prime number have no non-trivial automorphisms and so

Corollary: *For affine spaces of dimensions* ≥ 2 *over fields* $\mathbb{Q}$, $\mathbb{R}$ *and* Z_p p *prime number, every collineation is an affinity.* (**Remark:** The special case for the *universe* (the three dimensional real affine space) goes back to EULER.)