MA-219 Linear Algebra

14 A. Determinants – Rules for computation

October 28, 2003 ; Submit solutions **before 11:00 AM ; November 10, 2003.**

14.5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $n \times n$-matrices over the field K.

a). If both $\mathfrak{A}$ and $\mathfrak{B}$ are invertible, then :

(1) $\operatorname{Adj}(\mathfrak{A}\mathfrak{B}) = \operatorname{Adj}\mathfrak{B} \cdot \operatorname{Adj}\mathfrak{A}$. (2) $\operatorname{Adj}\mathfrak{A}^{-1} = (\operatorname{Adj}\mathfrak{A})^{-1}$.

(3) $\operatorname{Det}(\operatorname{Adj}\mathfrak{A}) = (\operatorname{Det}\mathfrak{A})^{n-1}$. (4) $\operatorname{Adj}(\operatorname{Adj}\mathfrak{A}) = (\operatorname{Det}\mathfrak{A})^{n-2}\,\mathfrak{A}$.

(**Remark :** All these formulas, other than (2), are also valid for not-invertible matrices (for (4) assume $n > 1$).)

b). Suppose that $\mathfrak{A}$ is not-invertible. Then the rank of the adjoint matrix $\operatorname{Adj}\mathfrak{A}$ is 1 if $\operatorname{Rank}\mathfrak{A} = n-1$ and is 0 if $\operatorname{Rank}\mathfrak{A} < n-1$.

14.6. Let K be a field.

a). Let $f_1, \ldots, f_n$ functions on the set D with values in the field K. Then $f_1, \ldots, f_n$ are linearly independent in K^D if and only if the function

$$(t_1, \ldots, t_n) \longmapsto \begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}$$

on D^n is not the zero-function. (**Remark :** See T9.4 – Determinants of this form are called alternant or (particularly in Physics) Slater's Determinant. For example the Vandermonde's determinant corresponding to $f_i := t^{i-1}$, $i = 1, \ldots, n$, $D := K$, see the part c) below and the Cauchy's double-alternants, see the part d) below.)

b). Let $f_1, \ldots, f_n$ be polynomial functions over K of deg $< n-1$, $n \in \mathbb{N}^*$. For all $t_1, \ldots, t_n \in K$, prove that

$$\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} = 0\,.$$

c). (Vandermonde's determinant) For elements $a_0, \ldots, a_n \in K$,

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = \prod_{0 \le i < j \le n} (a_j - a_i)\,.$$

(**Hint :** Induction on n. – See also exercise 13.2c).)

d). (Cauchy's Double-alternant) Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$ with $a_i + b_j \neq 0$ for all $i, j = 1, \ldots, n$. Then

$$\operatorname{Det}\left(\left(\frac{1}{a_i + b_j}\right)_{1 \le i,j \le n}\right) = \frac{\prod_{1\le i<j\le n}(a_j - a_i)\ \prod_{1\le i<j\le n}(b_j - b_i)}{\prod_{i,j=1}^{n}(a_i + b_j)}\,.$$

(**Hint :** Induction on n. – See also 13.2-d).)

e). For $t_1, \ldots, t_n, u_1, \ldots, u_n \in \mathbb{C}$, compute

$$\begin{vmatrix} \sin(t_1 + u_1) & \sin(t_1 + u_2) & \cdots & \sin(t_1 + u_n) \\ \sin(t_2 + u_1) & \sin(t_2 + u_2) & \cdots & \sin(t_2 + u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(t_n + u_1) & \sin(t_n + u_2) & \cdots & \sin(t_n + u_n) \end{vmatrix}.$$

(**Hint**: The two cases $n \leq 2$ and $n > 2$ seperately. See. also Aufg. 17.)

14.7. a). Let $P_i = (a_{1i}, \ldots, a_{ni})$, $i = 0, \ldots, n$ be points in the affine space $\mathbb{A}^n(K) = K^n$. Then the P_i are affinely dependent if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0 .$$

b). Let $P_i = (a_{1i}, \ldots, a_{ni})$, $i = 1, \ldots, n$ be affinely independent points in $\mathbb{A}^n(K) = K^n$. The equation of the affine hyperplane H in $\mathbb{A}^n(K)$ generated by the points $P_1, \ldots, P_n$ is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0 ,$$

i.e. the point $P = (x_1, \ldots, x_n) \in K^n$ belong to H if and only if its component satisfy the above (affine) equation. (See exercise 13.6-b).)

14.8. Let $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$ be elements of a field K. For the determinant

$$D_k := \begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{k-1} & b_{k-1} \\ 0 & 0 & 0 & \cdots & c_{k-1} & a_k \end{vmatrix} ,$$

prove the recursion formula $D_0 = 1$, $D_1 = a_1$, $D_k = a_k D_{k-1} - b_{k-1} c_{k-1} D_{k-2}$, for $k = 2, \ldots, n$.

14.9. Prove the following determinant formulas for the $n \times n$-matrices over a field K:

a).

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{vmatrix} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} a^{n-2k} b^{2k} .$$

For $a = 2$, $b = 1$ and for $a = b = 1$ compute the value of this determinant directly and verify this with the given sum-formula.

b).

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ -b & a & b & \cdots & 0 & 0 \\ 0 & -b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & -b & a \end{vmatrix} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} a^{n-2k} b^{2k} .$$

For $a = b = 1$, this determinant is the Fibonacci-number [1]) f_{n+1}.

c).

$$\begin{vmatrix} a_0+a_1 & a_1 & 0 & \cdots & 0 \\ a_1 & a_1+a_2 & a_2 & \cdots & 0 \\ 0 & a_2 & a_2+a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}+a_n \end{vmatrix} = \sum_{k=0}^{n} \Big(\prod_{i\neq k} a_i \Big).$$

d).

$$\begin{vmatrix} \cos\varphi & 1 & 0 & \cdots & 0 \\ 1 & 2\cos\varphi & 1 & \cdots & 0 \\ 0 & 1 & 2\cos\varphi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\varphi \end{vmatrix} = \cos n\varphi\,, \quad \varphi \in \mathbb{C}.$$

(**Remark:** See the recursion for the modified Tchebychev Polynomial $\widetilde{T}_n$. [2]))

On the other side one can see (simple) test-exercises; their solutions need not be submitted.

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[1]) **Fibonacci-numbers.** The sequence $(f_n)_{n\in\mathbb{N}}$ of integers which is defined recursively as: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ is called the Fibonacci sequence and its n-th term f_n is called the n-th Fibonacci number. First few terms of the Fibonacci sequence are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... For the n-th Fibonacci number there is an explicit formula:

(Binet's formula): $f_n := \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$

[2]) **Tchebychev Polynomials.** For $n \in \mathbb{N}$ the polynomials

$$T_n(X) := \sum_{k=0}^{[\frac{n}{2}]} \left(-\frac{1}{4}\right)^k \frac{n}{n-k}\binom{n-k}{k} X^{n-2k} \quad \text{and} \quad U_n(X) := \sum_{k=0}^{[\frac{n}{2}]} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} X^{n-2k}$$

are called Tchebychev polynomials of first and second kind respectively.

Properties of Tchebychev polynomials.

1). $T_0 = 2$, $T_1 = X$ and $T_{n+2} = XT_{n+1} - \frac{1}{4}T_n$ for every $n \in \mathbb{N}$.

2). $2^{n-1}T_n(\cos(\varphi)) = \cos(n\varphi)$ for every $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

3). For $n \in \mathbb{N}$, put $\widetilde{T}_n(X) := 2^{n-1}T_n(X)$. Then

(i) $\widetilde{T}_0 = 1$, $\widetilde{T}_1 = X$ and $\widetilde{T}_{n+2} = 2X\widetilde{T}_{n+1} - \widetilde{T}_n$ for every $n \in \mathbb{N}$.

(ii) Let $n \in \mathbb{N}$. Then $\widetilde{T}_n(1) = 1$, $\widetilde{T}_n(-1) = (-1)^n$ and $\widetilde{T}_n(0) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $\widetilde{T}_n(\cos(\varphi)) = \cos(n\varphi)$ for every $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

4). T_n and $\widetilde{T}_n$ have n-distinct real zeros in the open interval $(-1, 1)$, namely: $\cos((2k+1)\pi/2n)$ for $k = 0, \ldots, n-1$ and therefore $T_n(X) = \prod_{k=0}^{n-1}(X - \cos((2k+1)\pi/2n)$ for every $n \geq 1$.

5). $U_0 = 1$, $U_1 = X$ and $U_{n+2} = XU_{n+1} -$ ı4𝔘n for every $n \in \mathbb{N}$.

6). $2^{n-1}U_{n-1}(\cos(\varphi)) = \sin(\mathrm{n}\varphi)\sin(\varphi)$ for every $n \in \mathbb{N}^+$ and $\varphi \in \mathbb{R}$, with $\varphi \notin \mathbb{Z}\pi$.

7). Let $n \in \mathbb{N}$. Then $U_n(X) = \prod_{k=1}^{n}(X - \cos((k\pi)/(n+1))$ and $U_{2n}(X) = \prod_{k=1}^{n}(X^2 - \cos^2((k\pi)/(2n+1))$. In particular, $n+1 = 2^nU_n(1) = 2^n \cdot \prod_{k=1}^{n}(1 - \cos((k\pi)/(n+1))$ and $2n+1 = 2^{2n}U_{2n}(1) = 2^{2n} \cdot \prod_{k=1}^{n}(1 - \sin^2((k\pi)/(2n+1))$ for every $n \geq 1$.

Test-Exercises

T14.28. Determine for which $a \in \mathbb{R}$ the following system of linear equations over $\mathbb{R}$ have exactly one solution and in this case find the solution by using Cramer's rule:

$$\begin{aligned} ax_1 + x_2 + x_3 &= b_1 \\ x_1 + ax_2 + x_3 &= b_2 \\ x_1 + x_2 + ax_3 &= b_3 \,, \end{aligned} \qquad \begin{aligned} x_1 + x_2 - x_2 &= b_1 \\ 2x_1 + 3x_2 + ax_2 &= b_2 \\ x_1 + ax_2 + 3x_2 &= b_3 \,. \end{aligned}$$

T14.29. Let $\mathfrak{A} = (a_{ij})$ be a $n \times n$-matrix over the field K. For $c_1, \ldots, c_n \in K^\times$ prove that $\operatorname{Det}(a_{ij}) = \operatorname{Det}(c_i c_j^{-1} a_{ij})$. In particular, $\operatorname{Det}(a_{ij}) = \operatorname{Det}\big((-1)^{i+j} a_{ij}\big)$.

T14.30. Let $\mathfrak{A}' = (a'_{ij})$ be the $n \times n$-matrix obtained from the $n \times n$-matrix $\mathfrak{A} = (a_{ij})$ by reflecting the entries along the main-diagonal, i.e. $a'_{ij} = a_{n-j+1,\, n-i+1}$. Then $\operatorname{Det}\mathfrak{A}' = \operatorname{Det}\mathfrak{A}$.

T14.31. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $n \times n$-matrices with the columns $x_1, \ldots, x_n$ resp. $y_1, \ldots, y_n$. For a subset $J \subseteq \{1, \ldots, n\}$, let $\mathfrak{C}_J$ denote the $n \times n$-matrix with columns $z_1^{(J)}, \ldots, z_n^{(J)}$, where

$$z_i^{(J)} := \begin{cases} x_i \,, & \text{if } i \in J \,, \\ y_i \,, & \text{if } i \notin J \,. \end{cases}$$

Then

$$\operatorname{Det}(\mathfrak{A} + \mathfrak{B}) = \sum_{J \subseteq \{1,\ldots,n\}} \operatorname{Det}\mathfrak{C}_J \,.$$

(**Hint:** $\operatorname{Det}(\mathfrak{A} + \mathfrak{B}) = \Delta_{\mathfrak{e}}(x_1 + y_1, \ldots, x_n + y_n) = \sum_J \Delta_{\mathfrak{e}}(z_1^{(J)}, \ldots, z_n^{(J)}) = \sum_J \operatorname{Det}\mathfrak{C}_J$.)

T14.32. **a).** Suppose that a column (or a row) of the $n \times n$-matrix $\mathfrak{A}$ has all entries 1. For the cofactors $(-1)^{i+j} A_{ij}$, $i, j = 1, \ldots, n$ of $\mathfrak{A}$, prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} A_{ij} = \operatorname{Det}\mathfrak{A} \,.$$

b). Let $\mathfrak{A} = (a_{ij})$ be a $n \times n$-matrix over the field K with the cofaktorsn $(-1)^{i+j} A_{ij}$, $i, j = 1, \ldots, n$. Further, let

$$\mathfrak{I} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathrm{M}_n(K)$$

be the matrix with all coefficients are 1. Then

$$\operatorname{Det}(\mathfrak{A} + a\mathfrak{I}) = \operatorname{Det}\mathfrak{A} + a \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \,.$$

T14.33. Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(\mathbb{Q})$ be an invertible matrix with integer coefficients a_{ij}. Then all the coefficients of the inverse matrix $\mathfrak{A}^{-1}$ are integers if and only if $\operatorname{Det}\mathfrak{A} = \pm 1$.

T14.34. Let $\mathfrak{A} \in \mathrm{M}_n(K)$ be an upper triangular. Then $\operatorname{Adj}\mathfrak{A}$ is also upper triangular and if $\mathfrak{A}$ is invertible, then $\mathfrak{A}^{-1}$ is also upper triangular.

T14.35. Let f_{ij}, $i, j = 1, \ldots, n$ be differentiable functions on $D \subseteq \mathbb{K}$. Then

$$\begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix}' = \begin{vmatrix} f'_{11} & \cdots & f'_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f'_{21} & \cdots & f'_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f'_{n1} & \cdots & f'_{nn} \end{vmatrix} .$$

T14.36. If $\sigma \in \mathfrak{S}(I)$ is a permutation of the finite indexed I and let

$$\mathfrak{P}_\sigma = (\delta_{i\sigma(j)}) \in \mathrm{M}_I(K)$$

be the permutation matrix associated to σ. This is the matrix obtained from the unit matrix $\mathfrak{E}_I$ by permuting the columns according to σ : The j-th column of $\mathfrak{P}_\sigma$ is $e_{\sigma(j)}$, see also T13.23. Then for $\sigma, \tau \in \mathfrak{S}(I)$:

a). $\operatorname{Det} \mathfrak{P}_\sigma = \operatorname{Sign} \sigma$. **b).** $\mathfrak{P}_{\sigma\tau} = \mathfrak{P}_\sigma \mathfrak{P}_\tau$. **c).** $(\mathfrak{P}_\sigma)^{-1} = \mathfrak{P}_{\sigma^{-1}} = {}^t(\mathfrak{P}_\sigma)$.

T14.37. Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_I(K)$ be a skew-symmetric matrix (I finite indexed), i.e. ${}^t\mathfrak{A} = -\mathfrak{A}$. If $|I|$ is odd and if $\operatorname{Char} K \neq 2$, i.e. $2 = 2 \cdot 1_K \neq 0$ in K, then $\operatorname{Det} \mathfrak{A} = 0$.

T14.38. Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ with $|a_{ii}| > \sum_{j=1, j\neq i}^n |a_{ij}|$. Then $a_{11} \cdots a_{nn} \operatorname{Det} \mathfrak{A} > 0$. (**Hint:** By exercise 4.2 $\operatorname{Det} \mathfrak{A} \neq 0$ for such a matrix. Therefore the continuous polynomial function

$$f(t) := \begin{vmatrix} a_{11} & ta_{12} & ta_{13} & \cdots & ta_{1n} \\ ta_{21} & a_{22} & ta_{23} & \cdots & ta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ta_{n-1,1} & ta_{n-1,2} & ta_{n-1,3} & \cdots & ta_{n-1,n} \\ ta_{n1} & ta_{n2} & ta_{n3} & \cdots & a_{nn} \end{vmatrix}$$

has no zero in the interval $[0, 1]$ and so the values $f(0)$ und $f(1)$ havae the same sign.)

T14.39. Let K be a field and let $\mathfrak{A} \in \mathrm{M}_r(K)$, $\mathfrak{B} \in \mathrm{M}_s(K)$, $\mathfrak{C} \in \mathrm{M}_{r,s}(K)$. Then

$$\operatorname{Det} \begin{pmatrix} \mathfrak{C} & \mathfrak{A} \\ \mathfrak{B} & 0 \end{pmatrix} = (-1)^{rs} \operatorname{Det} \mathfrak{A} \cdot \operatorname{Det} \mathfrak{B}.$$

T14.40. Let D be a set, $t_1, \ldots, t_n \in D$ and $f_0, \ldots, f_n$ be linearly independent K-valued functions on D such that the $(n+1) \times n$-matrix

$$\begin{pmatrix} f_0(t_1) & \cdots & f_0(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{pmatrix}$$

has the maximal rank n. (because of the linear independence of $f_0, \ldots, f_n$, this is the case in general, see. Aufg. 17. In this case we say that the points $t_1, \ldots, t_n$ are in general position with respect to the $f_0, \ldots, f_n$.) Then the function

$$t \longmapsto \begin{vmatrix} f_0(t) & f_0(t_1) & \cdots & f_0(t_n) \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}$$

upto a uniquely determined constant factor $\lambda \neq 0$, is a non-trivial linear combination of the functions $f_0, \ldots, f_n$, which vanish on the points $t_1, \ldots, t_n$.

T14.41. Let D be a set, $E := \{t_1, \ldots, t_n\}$ be a subset of D with n elements and let $f_1, \ldots, f_n$ K-valued functions on D with

$$\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} \neq 0.$$

The functions $f_1|E, \ldots, f_n|E$ form a basis of K^E. For arbitrary elements $b_1, \ldots, b_n \in K$, there exists a unique linear combination f of $f_1, \ldots, f_n$ with $f(t_i) = b_i$, $i = 1, \ldots, n$. This follows from the equation

$$\begin{vmatrix} f(t) & b_1 & \ldots & b_n \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} = 0$$

by expanding in terms of the first column. (**Remark:** The uniquely determined function f is called the solution of the interpolation problem $f(t_i) = b_i$, $i = 1, \ldots, n$ with the functions $f_1, \ldots, f_n$.)

T14.42. Let $P_1 = (a_{11}\,, a_{21})\,, P_2 = (a_{12}\,, a_{22})\,, P_3 = (a_{13}\,, a_{23})$ be three points in $\mathbb{R}^2$ which do not lie on a line. Then

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & a_{11} & a_{12} & a_{13} \\ x_2 & a_{21} & a_{22} & a_{23} \\ x_1^2+x_2^2 & a_{11}^2+a_{21}^2 & a_{12}^2+a_{22}^2 & a_{13}^2+a_{23}^2 \end{vmatrix} = 0$$

is the equation of the circle passing through P_1, P_2, P_3.

T14.43. Let (a_{ij}) and (b_{ij}) be two $n \times n$-matrices over the field K. Then:

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \cdots & b_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

T14.44. Compute the following $n \times n$-determinant over $\mathbb{Q}$:

$$\begin{vmatrix} 1 & n & n & \cdots & n \\ n & 2 & n & \cdots & n \\ n & n & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix},$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 2 & 3 & \cdots & n-1 \\ 3 & 2 & 1 & 2 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \end{vmatrix}.$$

T14.45. Verify the following determinant formulas for $(n+1) \times (n+1)$-matrices with coefficients in a field K. (At the places marked by $*$ one may take arbitrary elements of K.)

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix} = (a+nb)(a-b)^n\,, \quad \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ b_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ * & b_2 & a_2 & \cdots & a_2 & a_2 \\ * & * & b_3 & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & b_n & a_n \end{vmatrix} = (a_1-b_1)\cdots(a_n-b_n)\,,$$

$$\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 1 & a_1+b_1 & * & \cdots & * \\ 1 & a_1 & a_2+b_2 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix} = b_1 \cdots b_n\,,$$

$$\begin{vmatrix} -a_1 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & a_2 & \cdots & 0 & 0 \\ 0 & 0 & -a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_n & a_n \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = (-1)^n (n+1)\, a_1 \cdots a_n\,.$$

T14.46. Prove the following determinant formulas by induction:

$$\begin{vmatrix} a_1+b_1 & b_1 & b_1 & \cdots & b_1 \\ b_2 & a_2+b_2 & b_2 & \cdots & b_2 \\ b_3 & b_3 & a_3+b_3 & \cdots & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & b_n & \cdots & a_n+b_n \end{vmatrix} = a_1\cdots a_n + \sum_{k=1}^{n}\Big(\prod_{i\neq k} a_i\Big) b_k\ ,$$

$$\begin{vmatrix} x+a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix} = x^n + a_1x^{n-1} + \cdots + a_n\ ,$$

$$\begin{vmatrix} a_1 & \cdots & 0 & 0 & \cdots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & a_n & b_n & \cdots & 0 \\ 0 & \cdots & b_n & a_n & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ b_1 & \cdots & 0 & 0 & \cdots & a_1 \end{vmatrix} = \prod_{k=1}^{n}(a_k^2 - b_k^2)\ .$$

T14.47. Show that

$$\begin{vmatrix} 1^n & 2^n & 3^n & \cdots & (n+1)^n \\ 2^n & 3^n & 4^n & \cdots & (n+2)^n \\ 3^n & 4^n & 5^n & \cdots & (n+3)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n+1)^n & (n+2)^n & (n+3)^n & \cdots & (2n+1)^n \end{vmatrix} = (-1)^{\binom{n+1}{2}}\,(n!)^{n+1}\ .$$

(**Hint:** Since $(i+j-1)^n = \sum_{k=1}^{n+1}\binom{n}{k-1} i^{k-1}(j-1)^{n+1-k}$, the above matrix is the product of two matrices and their determinants can be computed by using the Vandermonde's determinant, see T14.6-c).)

T14.48. Suppose that the matrix $\mathfrak{A} = (a_{ij}) \in \mathrm{GL}_n(K)$ satisfy the hypothesis of eexercise 13.9 and suppose that $\mathfrak{A} = \mathfrak{L}\mathfrak{D}\mathfrak{R}'$ with a diagonal matrix $\mathfrak{D} = \mathrm{Diag}\,(a_1, \ldots, a_n)$ and a normalised lower resp. upper triangular matrix $\mathfrak{L}$ resp. $\mathfrak{R}'$. Then $a_k = D_k/D_{k-1}$, $k = 1, \ldots, n$, where $D_k = \mathrm{Det}\,(a_{ij})_{1\leq i,j\leq k}$ is the k-th principal minor of $\mathfrak{A}$, $k = 0, \ldots, n$. (Put $D_0 = 1$.)

T14.49. Let $n \in \mathbb{N}^*$ and let K be a field. The canonical exact sequence

$$1 \longrightarrow \mathrm{SL}_n(K) \longrightarrow \mathrm{GL}_n(K) \xrightarrow{\mathrm{Det}} K^\times \longrightarrow 1$$

is a weak-split. Further, it is strong-split if and only if the power-map $x \mapsto x^n$ is an automorphism of $K^\times$.
(**Hint:** See exercise 11.4.)

T14.50. (Tchebychev–Systems) Let K be a field, I be a set and let K^I be the algebra the K–valued functions on I. Further, let $f_1, \ldots, f_n \in K^I$.

a). The following statements are equivalent :

(1) $f_1, \ldots, f_n$ are linearly independent over K.
(2) There exist $j_1, \ldots, j_n \in I$ such that the matrix $\mathfrak{V}_{\mathbf{f}}(j_1, \ldots, j_n) := (f_r(j_s))_{1\leq r,s\leq n} \in \mathrm{M}_n(K)$ is invertible.
(3) There exist $j_1, \ldots, j_n \in I$ such that the determinant $\mathrm{V}_{\mathbf{f}}(j_1, \ldots, j_n) := \mathrm{Det}\,\mathfrak{V}_{\mathbf{f}}(j_1, \ldots, j_n) \neq 0$.
(**Hint:** See also exercise ???? — The matrices $\mathfrak{V}_{\mathbf{f}}(j_1, \ldots, j_n)$ resp. its determinants $\mathrm{V}_{\mathbf{f}}(j_1, \ldots, j_n)$ corresponding to the system of functions $\mathbf{f} = (f_1, \ldots, f_n)$ are called generalised Vandermond's matrices resp. determinants and are also called alternants. The usual Vandermond's matrices

and determinants (see T14.6-c)) correspond to the system of the polynomial functions $1, x, \ldots, x^{n-1}$ from K to itself.)

b). Suppose that $|I| \geq n$. The following statements are equivalent :

(1) For every subset $J \subseteq I$ with $|J| = n$, $f_1|J, \ldots, f_n|J$ is a basis of K^J .
(2) For every function $g \in K^I$ and every subset $J \subseteq I$ with $|J| = n$, there exists a unique n–tuple $(b_1, \ldots, b_n) \in K^n$ such that $g(j) = \sum_{r=1}^n b_r f_r(j)$ for all $j \in J$.
(3) For $b_1, \ldots, b_n \in K$, if the function $\sum_{r=1}^n b_r f_r$ has n distinct zeros on I , then $b_1 = \cdots = b_n = 0$.
(4) For distinct elements $j_1, \ldots, j_n \in I$, the generalised Vandermond's matrix $\mathfrak{V}_{\mathbf{f}}(j_1, \ldots, j_n)$ is invertible.
(5) For distinct elements $j_1, \ldots, j_n \in I$, the generalised Vandermond's determinant $V_{\mathbf{f}}(j_1, \ldots, j_n) \neq 0$.

c). A system $\mathbf{f} = (f_1, \ldots, f_n)$ of functions in K^I which satisfy these equivalent conditions is called a Tchebychev– System on I . Let $(f_1, \ldots, f_n)$ be a Tchebychev–system on I . Then
(1) $(f_1|I', \ldots, f_n|I')$ is also a Tchebychev–System on I' for every subset $I' \subseteq I$ with $|I'| \geq n$.
(2) If $g_1, \ldots, g_n : I \to K$ generate the same subspace as that generated by $f_1, \ldots, f_n$ in K^I , then $(g_1, \ldots, g_n)$ is also a Tchebychev–system on I .

d). Let $f_1, \ldots, f_n$ be a Tchebychev–system on I and let $g \in K^I$. Further, let $j_1, \ldots, j_n \in I$ be distinct elements. For the linear combination f of of the $f_1, \ldots, f_n$ with $f(j_s) = g(j_s)$ for $s = 1, \ldots, n$ we have

$$\begin{vmatrix} f & g(j_1) & \cdots & g(j_n) \\ f_1 & f_1(j_1) & \cdots & f_1(j_n) \\ \cdots & \cdots & \cdots & \cdots \\ f_n & f_n(j_1) & \cdots & f_n(j_n) \end{vmatrix} = 0\,,$$

and hence f is determined by expanding this determinant in terms of the first column. (**Remark :** We say that the function f is obtained from the function g by interpolation with the system $f_1, \ldots, f_n$ with (interpolation–)knots $j_1, \ldots, j_n$.)

e). Let I be a topological space and let $K = \mathbb{R}$. Let $\Delta_n(I)$ denote the set of all those tuples in I^n , which have at least two equal components. Suppose that $(j_1, \ldots, j_n) \in I^n \smallsetminus \Delta_n(I)$ and an odd permutation $\sigma \in \mathfrak{S}_n$ such that $(j_1, \ldots, j_n)$ and $(j_{\sigma 1}, \ldots, j_{\sigma n})$ belong to the same connected component of $I^n \smallsetminus \Delta_n(I)$. Then there is no Tchebychev–system $(f_1, \ldots, f_n)$ on I , where $f_r : I \to \mathbb{R}, \ \nu = 1, \ldots, n$ are *continuous* functions.

T14.51. **a).** Let K be a field with at least n elements, $n \in \mathbb{N}^*$. Then the polynomial functions $1, x, \ldots, x^{n-1}$ form a Tchebychev–system on K . (this follows from ???.) More generally: If I is a set and $f : I \to K$ is injective, then the powers $1, f, \ldots, f^{n-1}$ for every $n \leq |I|$ form a Tcheychev–system on I.

b). Let K be a field with at least $2n$ elements, $n \in \mathbb{N}^*$ and $a_1, \ldots, a_n$ distinct elements in K. Then the rational functions $f_1(x) = 1/(x - a_1), \ldots, f_n(x) = 1/(x - a_n)$ form a Tchebychev–aystem on $K \smallsetminus \{a_1, \ldots, a_n\}$. (**Hint :** Consider $ff_1, \ldots, ff_n$ with $f(x) = (x - a_1) \cdots (x - a_n)$.)

c). The functions $1, \cos t, \ldots, \cos(n-1)t, \ n \in \mathbb{N}^*$ form a Tchebychev–system on the real interval $[0, \pi]$. (**Remark :** $1, \cos t, \ldots, \cos(n-1)t$ resp. $1, \cos t, \ldots, \cos^{n-1} t$ generate the same function space, see §27, Aufgabe 5c).)

d). The functions $\sin t, \ldots, \sin nt, \ n \in \mathbb{N}^*$ form a Tchebychev–system on the opne real interval $(0, \pi)$. (**Remark :** $\sin t, \ldots, \sin nt$ resp. $\sin t, \sin t \cos t, \ldots, \sin t \cos^{n-1} t$ generate the same function space.)

e). Let $n \in \mathbb{N}$. The $2n + 1$ functions $\exp(\mathrm{i}\nu t), \ \nu = -n, \ldots, -1, 0, 1, \ldots, n$ form a Tchebychv–system on the half-open real interval $[0, 2\pi)$, similarly the functions $1, \cos t, \sin t, \ldots, \cos nt, \sin nt$. (**Remark :** The given system is also a Tchebychev–systeme on the unit circle S^1 . Does there exists a Tchebychev–systeme with $2n + 2$ *continuous* functions $S^1 \to \mathbb{R}$ on the unit circle? See. Aufgabe 11d).)

T14.52. The space $\mathrm{M}_{m,n}(\mathbb{R})$ of real $(m \times n)$ –matrices has a natural toplogy which is defined by the metric $d((a_{ij}), (b_{ij})) := \mathrm{Max}\{|b_{ij} - a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$.

a). The determinant map $\mathrm{Det} : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous.

b). The set $\mathrm{GL}_n(\mathbb{R})$ of all invertible matrices in $\mathrm{M}_n(\mathbb{R})$ is open in $\mathrm{M}_n(\mathbb{R})$.

c). Let $r \in \mathbb{N}$. The set of all matrices of rank $\geq r$ in $\mathrm{M}_{m,n}(\mathbb{R})$ is open in $\mathrm{M}_{m,n}(\mathbb{R})$.

d). The set of all matrices of maximal rank $\mathrm{Min}\{m, n\}$ in $\mathrm{M}_{m,n}(\mathbb{R})$ is open in $\mathrm{M}_{m,n}(\mathbb{R})$.

T14.53. Let U be an open subset in $\mathbb{R}^n$ and let $g : U \to \mathbb{R}^m$ be a continuously differentiable map, i.e. the functions $g_i := p_i g, \ i = 1, \ldots, m$, where $p_i : \mathbb{R}^m \to \mathbb{R}$ are the canonical projections on the components, are partial differentiable with continuous partial derivatives $\partial_j g_i = \partial g_i / \partial t_j$, $1 \le i \le m$, $1 \le j \le n$. For $t \in U$, the matrix

$$\mathfrak{J}(g)(t) := \begin{pmatrix} \partial_1 g_1(t), & \cdots, & \partial_n g_1(t) \\ \vdots & \ddots & \vdots \\ \partial_1 g_m(t), & \cdots, & \partial_n g_m(t) \end{pmatrix}$$

is called the functional– or Jacobian–matrix of g in t, in the case $m = n$ its determinant $\mathrm{J}(g)(t) := |\mathfrak{J}(g)(t)|$ is called the functional– or Jacobian–determinant of g in t.

a). Suppose that $m = n$. Then the function $t \mapsto \mathrm{J}(g)(t)$ is continuous on U.

b). Suppose that $m = n$. The subset $\{t \in U \mid \mathfrak{J}(g)(t)$ is invertible $\}$ is open in U.

c). Let $r \in \mathbb{N}$. The subset $\{t \in U \mid \operatorname{rank} \mathfrak{J}(g)(t) \ge r\}$ is open in U.

d). The subset $\{t \in U \mid \mathfrak{J}(g)(t)$ has a maximal rank $\operatorname{Min}\{m, n\}\}$ is open in U. (**Remark:** In this case we say that g is regular at such a point.)

T14.54. **a).** The map $t \mapsto (1/t_1^2 + \cdots + t_n^2) \cdot t$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n \smallsetminus \{0\}$, has the functional determinant $-(1/(t_1^2 + \cdots + t_n^2)^n)$. (Man benutze §47, Aufgabe 12.)

b). (Polar coordinates) For the map $g : t \mapsto (g_1(t), \ldots, g_n(t))$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $n \ge 2$, where

$$\begin{aligned} g_1(t) &= t_1 \cos t_n \cdots \cos t_3 \cos t_2 \\ g_2(t) &= t_1 \cos t_n \cdots \cos t_3 \sin t_2 \\ g_3(t) &= t_1 \cos t_n \cdots \sin t_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ g_{n-1}(t) &= t_1 \cos t_n \sin t_{n-1} \\ g_n(t) &= t_1 \sin t_n \end{aligned}$$

we have : $\mathrm{J}(g)(t) = t_1^{n-1} \cos^{n-2} t_n \cdots \cos t_3$. (**Hint:** Induction on n.)