MA-219 Linear Algebra

14 B. Determinants – Orientation; Volume computation

November 03, 2003 ; Submit solutions **before 11:00 AM ; November 17, 2003.**

14.10. (Norm) Let A be a finite dimensional K-algebra. For $x \in A$, let $\lambda_x : A \to A$ denote the left multiplication $y \mapsto xy$ by x in A. Then λ_x is K-linear operator on A. Its determinant is called the *norm of* x (over K) and is denoted by $\mathrm{N}_K^A(x) = \mathrm{N}(x)$.

a). For all $x, y \in A$ and for all $a \in K$, show that

(1) $\mathrm{N}(xy) = \mathrm{N}(x)\mathrm{N}(y)$. (2) $\mathrm{N}(a) := \mathrm{N}(a \cdot 1_A) = a^n$, $n := \mathrm{Dim}_K A$.
(3) An element $z \in A$ is a unit in A if and only if $\mathrm{N}(x) \neq 0$ in K.

b). For all elements z of the $\mathbb{R}$-algebra $\mathbb{C}$, show that $\mathrm{N}_{\mathbb{R}}^{\mathbb{C}}(z) = |z|^2$. (Vgl. Aufg. 4.)

c). Let $A = \mathrm{M}_n(K)$ be the algebra the $n \times n$-matrices over the field K. For all $\mathfrak{A} \in A$, prove that $\mathrm{N}_K^A(\mathfrak{A}) = (\mathrm{Det}\,\mathfrak{A})^n$

14.11. Determine which of the following affinities of n-dimensional orientated real affine spaces preserve the orientation:

a). Point-reflections. **b).** Reflections of a hyperplane along a line resp. product of such r reflections, $r \in \mathbb{N}$. **c).** Shearings. **d).** Dilatations. **e).** Magnifications.

14.12. Show that the volume of the ellipsoid $\Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \;\Big|\; \dfrac{x_1^2}{a_1^2} + \cdots + \dfrac{x_n^2}{a_n^2} \leq 1\Big\} \subseteq \mathbb{R}^n$, $a_i \in \mathbb{R}_+^\times$, $1 \leq i \leq n$, is $\omega_n a_1 \cdots a_n$, where ω_n is the volume of the unit-sphere $\mathrm{S}(0,1) := \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \;\big|\; x_1^2 + \cdots + x_n^2 \leq 1\big\}$. (**Remark:** The ω_n can be computed by using integration and it is $\omega_n = \pi^{n/2}/(n/2)!$.)

14.13. Let $f_1, \ldots, f_n$ be a basis of the space of linear forms on $\mathbb{R}^n$. Let $\mathfrak{A} := (a_{ij}) \in \mathrm{GL}_n(\mathbb{R})$ be the transition matrix corresponding to the dual basis $e_1^*, \ldots, e_n^*$ of the standard basis. Then $f_j = \sum_{i=1}^n a_{ij} e_i^*$ and $f_1, \ldots, f_n$ is the dual basis corresponding to the basis $v_j = \sum_{i=1}^n b_{ij} e_i$, $j = 1, \ldots, n$, where $\mathfrak{B} := (b_{ij}) = {}^t\mathfrak{A}^{-1}$ is the contra-gradient matrix of $\mathfrak{A}$ (see 8.A, Aufg. 24). Let $d := |\mathrm{Det}\,\mathfrak{A}|$.

a). For $c_1, \ldots, c_n \geq 0$, the volume of $\big\{x \in \mathbb{R}^n \;\big|\; |f_i(x)| \leq c_i\,,\ i = 1, \ldots, n\big\}$ is $2^n c_1 \cdots c_n / d$.

b). For $c \geq 0$, the volume of $\big\{x \in \mathbb{R}^n \;\big|\; \sum_{i=1}^n |f_i(x)| \leq c\big\}$ is $2^n c^n / n!\, d$.

c). For $c \geq 0$, the volume of the ellipsoid $\big\{x \in \mathbb{R}^n \;\big|\; \sum_{i=1}^n |f_i(x)|^2 \leq c^2\big\}$ is $\omega_n c^n / d$, where ω_n is same as that in 14.12.

d). For $c_0, c_1, \ldots, c_n \in \mathbb{R}$ with $c_0 \leq c_1 + \cdots + c_n$, the volume of the simplex

$$\big\{x \in \mathbb{R}^n \;\big|\; f_i(x) \leq c_i\,,\ i = 1, \ldots, n\,,\ f_1(x) + \cdots + f_n(x) \geq c_0\big\}$$

is $b^n / n!\, d$, where $b := c_1 + \cdots + c_n - c_0$.

14.14. Let $n \in \mathbb{N}^*$. The group $\mathrm{GL}_n(\mathbb{R})$ is the direct product of the group $\mathrm{I}_n(\mathbb{R})$ of the volume preserving (or *unimodular*) matrices $\mathfrak{B} \in \mathrm{GL}_n(\mathbb{R})$ with $|\mathrm{Det}\mathfrak{B}| = 1$ and the group $\mathbb{R}_+^\times \mathfrak{E}_n \cong \mathbb{R}_+^\times$ of the scalar matrices $a\mathfrak{E}_n$, $a \in \mathbb{R}_+^\times$, i.e. every matrix $\mathfrak{A} \in \mathrm{GL}_n(\mathbb{R})$ has a representation $\mathfrak{A} = a\mathfrak{B} = \mathfrak{B}a$ with uniquely determined (by $\mathfrak{A}$) elements $a \in \mathbb{R}_+^\times$ and $\mathfrak{B} \in \mathrm{I}_n(\mathbb{R})$. (**Remark:** Every linear automorphism f of $\mathbb{R}^n$ is therefore composition of a volume preserving automorphism g and a magnification $a\,\mathrm{id}$ with positive magnification factor a, where g and a are uniquely determined by f. This g is called the *volume preserving part* and a is called the *magnification factor* of f.)

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

Test-Exercises

T14.55. Let V be a finite dimensional K-vector space. Compute the determinant of the following linear maps $f : V \to V$:

a). f is the homothecy $a\,\mathrm{id}_V$. **b).** f is a projection.

c). f is an involution (see exercises 12.5, 12.6 and 13.5).

d). f is a transvection or dilatation (see exercises 12.5, 12.6 and 13.5).

T14.56. Let $f : V \to V$ be a nilpotent endomorphism of the n-dimensional K-vector space V. Then show that $\mathrm{Det}\,(a\,\mathrm{id}_V + f) = a^n$ for all $a \in K$; More generally, $\mathrm{Det}\,(g + f) = \mathrm{Det}\,g$ for every operator g on V which commute with f. (See also ???.)

T14.57. Let $V := K[t]$ be the space of the polynomial functions over the field K with infinitely many elements and let $V_n := K[t]_n$ be the subspace of the polynomial functions of $\deg < n$, $n \in \mathbb{N}^*$.

a). For $a, b \in K$, let $\varepsilon : V \to V$ be defined by $f(t) \mapsto f(at+b)$. Show that ε is linear with $\varepsilon(V_n) \subseteq V_n$ for all n, and compute $\mathrm{Det}\,(\varepsilon|V_n)$.

b). Let $K = \mathbb{K}$. For $c_0, \ldots, c_r \in \mathbb{K}$, let $\delta : V \to V$ be the differential operator

$$f(t) \mapsto \sum_{k=0}^{r} c_k f^{(k)}(t)\,.$$

For $n \in \mathbb{N}^*$ show that δ is linear with $\delta(V_n) \subseteq V_n$ and compute $\mathrm{Det}\,(\delta|V_n)$.

T14.58. Let V be a finite dimensional $\mathbb{C}$-vector space and let $f : V \to V$ be a $\mathbb{C}$-linear operator on V. We consider V as a $\mathbb{R}$-vector space and so f is a $\mathbb{R}$-linear operator; its determinant is denoted by $\mathrm{Det}_{\mathbb{R}} f$.

Show that $\mathrm{Det}_{\mathbb{R}} f = |\mathrm{Det}\, f|^2$. (**Hint:** If $\mathfrak{A} + \mathrm{i}\,\mathfrak{B}$, $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(\mathbb{R})$ is the matrix of f with respect to the $\mathbb{C}$-basis $v_1, \ldots, v_n$ of V, then

$$\begin{pmatrix} \mathfrak{A} & -\mathfrak{B} \\ \mathfrak{B} & \mathfrak{A} \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R})$$

is the matrix of f with respect to the $\mathbb{R}$-basis $v_1, \ldots, v_n, \mathrm{i}v_1, \ldots, \mathrm{i}v_n$ and

$$\begin{vmatrix} \mathfrak{A} & -\mathfrak{B} \\ \mathfrak{B} & \mathfrak{A} \end{vmatrix} = \begin{vmatrix} \mathfrak{A} - \mathrm{i}\,\mathfrak{B} & -\mathfrak{B} \\ \mathfrak{B} + \mathrm{i}\,\mathfrak{A} & \mathfrak{A} \end{vmatrix} = \begin{vmatrix} \mathfrak{A} - \mathrm{i}\,\mathfrak{B} & -\mathfrak{B} \\ 0 & \mathfrak{A} + \mathrm{i}\,\mathfrak{B} \end{vmatrix}\,.$$

Special case: If A is a finite dimensional $\mathbb{C}$-algebra, then for all $x \in A$, we have $\mathrm{N}^A_{\mathbb{R}}(x) = |\mathrm{N}^A_{\mathbb{C}}(x)|^2$.)

T14.59. **a).** Let V be an orientated n-dimensional $\mathbb{R}$-vector space and let $\sigma \in \mathfrak{S}_n$ be a permutation. If the basis $v_1, \ldots, v_n$ represents the orientation of V, then the basis $v_{\sigma(1)}, \ldots, v_{\sigma(n)}$ of V represents the orientation of V if and only if σ is even. The basis $v_n, \ldots, v_1$ represents the orientation of V if and only if $n \equiv 0$ or $n \equiv 1$ modulo 4.

b). Let E be an orientated n-dimensinal $\mathbb{R}$-affine space. Suppose that the affine basis $P_0, \ldots, P_n$ of E represents the orientation of E. For a permutation $\sigma \in \mathfrak{S}(\{0, \ldots, n\})$, the affine basis $P_{\sigma(0)}, \ldots, P_{\sigma(n)}$ of E represents the orientation of E if and only if σ is even. The affine basis $P_n, \ldots, P_0$ of E represents the orientation of E if and only if $n \equiv 0$ or $n \equiv 3$ modulo 4.

T14.60. In every subgroup of the affine group $\mathrm{A}(E)$ of an orientated finite dimensional real affine space E which has at least one element which does not preserve the orientation, the orientation preserving maps form a subgroup of index 2.

T14.61. Suppose that the finite dimensional $\mathbb{R}$-vector space V is the direct sum of the subspaces U and W. By the following prescripion the orientation of any two spaces of U, V, W define a orientation of the third: If $\mathfrak{u} = (u_1, \ldots, u_r)$ resp. $\mathfrak{w} = (w_1, \ldots, w_s)$ bases of U resp. W, then the basis $(u_1, \ldots, u_r, w_1, \ldots, w_s)$ representation of the orientation of $V = U \oplus W$ if and only if the bases $\mathfrak{u}$ resp. $\mathfrak{w}$ both represent resp. donot represent both the orientations of U resp. W. (**Remark:** Note the dependence on the succession of U and W.)

T14.62. Let V be a finite dimensional $\mathbb{R}$-vector space, V' be a subspace and let $\overline{V} = V/V'$. By the following prescripion the orientation of any two spaces of $V', V, \overline{V}$ define a orientation of the third: If $v'_1, \ldots, v'_r \in V'$ is a basis of V' and the residue classes of $v_1, \ldots, v_s \in V$ is a basis of $\overline{V}$, then the basis $v'_1, \ldots, v'_r, v_1, \ldots, v_s$ of V represents the orientation of V if and only if the bases $v'_1, \ldots, v'_r$ of V' and $\overline{v}_1, \ldots, \overline{v}_s$ of $\overline{V}$ represent resp. donot represent both the orientation of V' resp. $\overline{V}$.

T14.63. Determine which of the following bases of $\mathbb{R}^n$ represent the standard orientation:

a). $n = 2$; $v_1 = (1, 1)$, $v_2 = (1, -1)$.

b). $n = 3$; $v_1 = (-1, 0, 1)$, $v_2 = (0, -1, 1)$, $v_3 = (1, -1, 1)$.

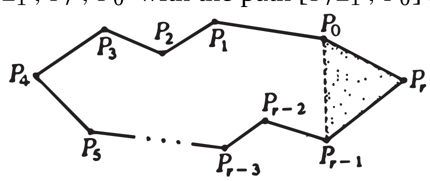
c). $n = 4$; $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 1, 1)$, $v_3 = (1, 1, 3, 1)$, $v_4 = (1, 1, 1, 4)$.

T14.64. Every $\mathbb{C}$-linear isomorphism of a finite dimensional complex vector space is orientation preserving.

T14.65. Let E be a real affine plane with the volume function $\lambda_{\mathfrak{v}}$ with respect to the basis v_1, v_2 of the space of the translations of E and $P_0, \ldots, P_r$, $r \geq 2$, be the points with coordinates (a_j, b_j), $j = 0, \ldots, r$ with respect to an affine coordinate system $O; v_1, v_2$. Let $[P_0, P_1, \ldots, P_r, P_0]$ be a simple closed polgonal path, i.e. the edges intersects if and only if they are consecutive and intersect only in vertices. Show that the surface-area of the closed polygon, upto sign, is equal to

$$\frac{1}{2}\left(\mathrm{Det}\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} + \cdots + \mathrm{Det}\begin{pmatrix} a_{r-1} & a_r \\ b_{r-1} & b_r \end{pmatrix} + \mathrm{Det}\begin{pmatrix} a_r & a_0 \\ b_r & b_0 \end{pmatrix}\right).$$

(**Hint:** What does the sign mean? One can think on the orientation of E. – Use induction and from the case $r-1$ to the case r use after renumbering the vertices of the polygon with vertices $P_0, \ldots, P_{r-1}$ and the triangle with the vertices P_{r-1}, P_r, P_0 with the path $[P_{r-1}, P_0]$ in common.)



T14.66. Draw the picture of the set $M := H_1 \cap H_2 \cap H_3$ in $\mathbb{R}^2$ with $H_i := \{(x, y) \in \mathbb{R}^2 \mid f_i(x, y) \geq 0\}$, $i = 1, 2, 3$ and $f_1(x, y) := x + 3y + 1$, $f_2(x, y) := -5x + y + 1$, $f_3(x, y) := x - y + 3$ and compute its surface-area.

T14.67. Let $P_0, \ldots, P_n \in \mathbb{R}^n$ be affinely independent points and S is the (convex) simplex with these vertices. Further, let $y_0, \ldots, y_n \in \mathbb{R}_+$ and H is the affine hyperplane in $\mathbb{R}^{n+1}$ passing through the points $(P_0, y_0), \ldots, (P_n, y_n) \in \mathbb{R}^{n+1}$. Then H is the graph of the affine function $h : \mathbb{R}^n \to \mathbb{R}$ with $h(P_i) = y_i$, $i = 0, \ldots, n$. If $T \subseteq \mathbb{R}^{n+1}$ is the region in between S and H, i.e.

$$T := \{(x, y) \in \mathbb{R}^{n+1} \mid x \in S,\ 0 \leq y \leq h(x)\},$$

then

$$\lambda^{n+1}(T) = \frac{y_0 + \cdots + y_n}{n+1}\,\lambda^n(S).$$

($\lambda^{n+1}(T)$ is additive in $(y_0, \ldots, y_n)$ and does not alter even if the values $y_0, \ldots, y_n$ are permutated. One may therefore assume that all y_i are equal or that all y_i other than one value y_{i_0} are zero.) Compute the volume of the following region in $\mathbb{R}^3$, where the plane surface is: