MA-219 Linear Algebra

Test 2

Sunday, October 05, 2003 ; 10:00 AM - 11:30 AM

- Wherever necessary justify hints and your answers.

T2.1. a). In the $\mathbb{R}$-vector space $V := \mathbb{R}^4$, find a complement W of the $\mathbb{R}$-subspace

$$U := \left\{ (a, b, c, d) \in V \mid \int_0^1 t(a + bt + ct^2 + dt^3)dt = 0 \right\}.$$

and the projection p of V onto U along W. [3 Points]

b). Let $a, b, c \in \mathbb{R}$ and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the $\mathbb{R}$-linear map defined by $f(e_1) = be_1 + ce_2$, $f(e_2) = -ae_1 + ce_3$ and $f(e_3) = -ae_2 - be_3$, where $e_1, e_2, e_3 \in \mathbb{R}^3$ is the standard $\mathbb{R}$-basis of $\mathbb{R}^3$. Find the rank(f). [3 Points]

c). Let V be a infinite dimensional vector space over a field K with basis $\{v_i \mid i \in I\}$ and let $\{v_i^* \mid i \in I\}$ be the coordinate functions with respect to the basis $\{v_i \mid i \in I\}$. Show that $\sum_{i\in I} K v_i^* \neq V^*$. [4 Points]

T2.2. Let U be a K-subspace of a finite dimensional vector space V over a field K.

a). Suppose that $\frac{1}{2}\,\mathrm{Dim}_K\, V \leq \mathrm{Dim}_K\, U < \mathrm{Dim}_K\, V$. Show that there exist complements W_1, W_2 of U in V such that $W_1 \cap W_2 = 0$. [5 Points]

b). Show that U has a unique complement in V if and only if either $U = 0$ or $U = V$. [5 Points]

T2.3. a). Let V be a finite dimensional vector space over a field K and let $f \in \mathrm{End}_K(V)$. Show that there exists $g \in \mathrm{Aut}_K(V)$ and projections p, q of V such that $f = pg = gq$. [5 Points]

b). Let K be a field and let $s, n \in \mathbb{N}$, $s \leq n$. Show that every affine K-subspace of dimension s of K^n is a solution set of a linear system consisting $n - s$ equations in n unknowns of rank $n - s$. [5 Points]

T2.4. Let V be a vector space over a field K.

a). Let $f_1, \ldots, f_n \in V^*$ be linear forms on in V and let $f : V \to K^n$ be the homomorphism defined by $f(x) := (f_1(x), \ldots, f_n(x))$. Show that $\mathrm{Dim}_K\, (Kf_1 + \cdots + Kf_n) = \mathrm{rank}\ f$. [5 Points]

b). Let $x_1, \ldots, x_n \in V$ be all non-zero vectors. Assume that K has at least n elements. Show that there exists a hyperplane H in V such that $x_i \notin H$ for all $i = 1, \ldots, n$. [6 Points]