MA 315 Galois Theory / January-April 2014

(Int PhD. and Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** MA LH-4 (**if LH-1 is not free**) / LH-1

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Midterms : Thu. Feb 27 (14:00-16:30); **Quizzes : (Wed-Lect)** Feb 05 ; Mar 12 ; **(Sat-Lect)** April 05
Final Examination : 9 AM-12 Noon, Thursday, April 24, 2014

Evaluation Weightage : Quizzes : 10% **Seminar :** 10% **Midterms :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

MID TERM

Thursday, February 27, 2014 **14:00 to 16:30** **Maximum Points :** 50 Points

• Question T1.6 is C O M P U L S O R Y. • Attempt O N L Y F I V E Questions.

T1.1 Let $K\,|\,k$ be a field extension and let $x, y \in K$.

(a) If $x \in K$ is algebraic over k with $\deg\,\mu_{x,k}$ is odd, then show that $\deg\,\mu_{x^2,k}$ is also odd and $k(x) = k(x^2)$. [3 points]

(b) If $x, y \in K$ are algebraic over k, then $[k(x,y):k] \leq \deg\,\mu_{x,k} \cdot \deg\,\mu_{y,k}$. Moreover, if $\deg\,\mu_{x,k}$ and $\deg\,\mu_{y,k}$ are relatively prime i. e., $\gcd(\deg\mu_{x,k}, \deg\mu_{y,k}) = 1$, then the equality holds. [3 points]

(c) Suppose that $K\,|\,k$ is finite of degree m and $f \in k[X]$ is an irreducible polynomial over k of degree n. If $\gcd(m,n) = 1$, then show that f is also irreducible over K. (**Hint :** Let x be a zero of f in a field extension $L\,|\,K$. Then observe that $\deg f \leq [K(x):K]$.) [4 points]

T1.2 Let k be a field of characteristic $\neq 2$.

(a) Let $K\,|\,k$ be a field extension of degree $[K:k] = 2$. Show that $K = k(x)$ with $x^2 = a \in k$. Moreover, show that $K\,|\,k$ is Galois extension with Galois group $\operatorname{Gal}(K\,|\,k) \approx \mathbb{Z}^{\times}$. [3 points]

(b) Let $K\,|\,k$ be a field extension and let $x, y \in K$ with $x^2 = a \in k$ and $y^2 = b \in k$. Determine necessary and sufficient condition so that there exists a k-algebra isomorphism $k(x) \xrightarrow{\sim} k(y)$. (**Hint :** The required necessary and sufficient condition is there exists $c \in k^{\times}$ such that $b = c^2 a$. To verify this use the k-bases $1, x$ and $1, y$ of $k(x)$ and $k(y)$, respectively.) [5 points]

(c) Show that the $\mathbb{Q}$-vector spaces $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are isomorphic, but they are not isomorphic as fields. (**Hint :** Use the criterion in part (b).) [2 points]

T1.3 Let $K\,|\,k$ be a field extension.

(a) Show that the following statements are equivalent:

(i) $K\,|\,k$ is algebraic.

(ii) For every intermediary subfield $L \in \mathfrak{F}(K\,|\,k)$, every k-algebra homomorphism $\sigma : L \to L$ is an automorphism. [5 points]

(b) Suppose that $K\,|\,k$ is finite. Then show that $\#\operatorname{Gal}(K\,|\,k)$ divides $[K:k]$. (**Hint :** Use the fixed field $K^{\operatorname{Gal}(K|k)}$.) [5 points]

T1.4 Let $K = k(x)$ be a finite simple extension of a field k of degree n.

(a) Let L be an intermediary subfield of $K\,|\,k$ and let $\mu_{x,L} = b_0 + b_1X + \cdots + b_{m-1}X^{m-1} + X^m \in L[X]$ be the minimal polynomial of x over L. Show that $L = k(b_0, \ldots, b_{m-1})$. (**Hint :** Put $L' := k(b_0, \ldots, b_{m-1})$. Then $L' \subseteq L$, $L'(x) = L(x) = K$ and $\mu_{x,L} = \mu_{x,L'}$.) [5 points]

(b) Show that the number of intermediate fields L with $k \subseteq L \subseteq K$ is at most 2^{n-1}, i. e. $\#\mathfrak{F}(K\,|\,k) \leq 2^{[K:k]-1}$.(**Hint :** Use part (a).) Give an example to show that this inequality can be very strict. [5 points]

T1.5 Let $\mathbb{F}_q$ be a finite field with q elements.

(a) Let $f \in \mathbb{F}_q[X]$ be an irreducible polynomial of degree m. Show that the following statements are equivalent:

(i) f divides $X^{q^n} - X$ in $\mathbb{F}_q[X]$. (ii) f has a zero in $\mathbb{F}_{q^n}$. (iii) m divides n. [6 points]

(**Hint :** Use the fact that any two finite fields with the same cardinality are isomorphic as fields.)

(b) Let p, q be two distinct prime numbers. Verify that q is not a square in the field $\mathbb{Q}(\sqrt{p})$ and deduce that $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 4$. [4 points]

***T1.6** Let p be an odd prime number, $\zeta_p := e^{2\pi\mathrm{i}/p}$ and let $\mathbb{Q}^{(p)} := \mathbb{Q}(\zeta_p) \subseteq \mathbb{C}$. Show that

(a) $\mathbb{Q}(\cos(2\pi/p)) = \mathbb{R} \cap \mathbb{Q}^{(p)}$. [2 points]

(b) The minimal polynomial $\mu_{\zeta_p, \mathbb{Q}(\cos(2\pi/p))}$ is $X^2 - 2\cos(2\pi/p)X + 1$. [2 points]

(c) Find the degrees $[\mathbb{Q}^{(p)} : \mathbb{Q}(\cos(2\pi/p))]$ and $[\mathbb{Q}(\cos(2\pi/p)) : \mathbb{Q}]$. [3 points]

(d) The field extension $\mathbb{Q}^{(n)}\,|\,\mathbb{Q}(\cos(2\pi/p))$ is a Galois extension. Compute the Galois group $\mathrm{Gal}\,(\mathbb{Q}^{(p)}\,|\,\mathbb{Q}(\cos(2\pi/p)))$. [3 points]

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