MA-231 Topology

1. Sets and Maps

August 6, 2004 ; Submit solutions **before 10:00 AM ; August 16, 2004.**

1.1. Sketch the graphs of the following functions $\mathbb{R} \to \mathbb{R}$:

a). $x \mapsto \{x\} := x - [x]$ (sawtooth curve).

b). $x \mapsto \begin{cases} x - [x], & \text{if } [x] \le x < [x] + \frac{1}{2}, \\ [x] + 1 - x, & \text{if } [x] + \frac{1}{2} \le x < [x] + 1. \end{cases}$ (Distance to the next integer).

c). $x \mapsto \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$ (Heaviside† step-function)

d). $x \mapsto \lceil x \rceil :=$ the smallest integer $\ge x$. (Ceiling function)

e). $x \mapsto x|x| = (\text{Sign } x)x^2$.

f). $x \mapsto x + |x - 1|$.

g). $x \mapsto |x^2 - 4|$.

1.2. Let $f : X \to Y$ and let $g : Y \to X$ be maps. Show that

a). If gf is injective, then f is injective.

b). If gf is surjective, then g is surjective.

c). If gf is bijective, then f injective and g is surjective. (Give an example to show that neither f nor g is bijective even if gf is bijective.)

d). If gf is bijective and if f (resp. g) is bijective, then g (resp. f) is also bijektive.

1.3. a). Let $f : X \to Y$, $g : Y \to Z$ and let $h : Z \to W$ be maps. If gf and hg are bijective, then show that f, g and h are bijective.

b). Let $f : X \to Y$, $g : Y \to X$ and let $h : X \to Y$ be maps. From the equalities $gf = \mathrm{id}_X$ and $hg = \mathrm{id}_Y$, show that equality $f = h$, i.e. the bijectivity of g and $g^{-1} = f = h$.

1.4. The fibres of the real valued functions are also called level sets. Sketch the level sets of the following functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ corresponding to the values $-2, -1, 0, 1$ and 2:

a). $(x, y) \mapsto xy$. **b).** $(x, y) \mapsto |x - 1| + |y + 2|$. **c).** $(x, y) \mapsto |y - x|$.

d). $(x, y) \mapsto x^2 - 4x + y^2$. **e).** $(x, y) \mapsto \sqrt[3]{x + 1} - \sqrt{|y|}$. **f).** $(x, y) \mapsto xy - (x + y)$.

1.5. Let $f : X \to Y$ be a map and let $f_* : \mathfrak{P}(X) \to \mathfrak{P}(Y)$ (resp. $f^* : \mathfrak{P}(Y) \to \mathfrak{P}(X)$) be the map defined $f_*(A) := f(A)$, $A \subseteq X$ (resp. $f^*(B) := f^{-1}(B)$, $B \subseteq Y$).

a). The following are equivalent : (i) f is injective. (ii) f_* is injective. (iii) f^* is surjective.

b). The following are equivalent : (i) f is surjective. (ii) f_* is surjective. (iii) f^* is injective.

c). If f bijective, then so are f_* and f^* ; moreover, they are inverses of each other.

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

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† **Oliver Heaviside (1850-1925)** Oliver Heaviside was born on 18 May 1850 in Camden Town, London, England and died on 3 Feb 1925 in Torquay, Devon, England. Perhaps Heaviside has become more widely known due to the Andrew Lloyd Webber song Journey to the Heaviside Layer in the musical Cats, based on the poems of T S Eliot :

Up up up past the Russell hotel
Up up up to the Heaviside layer

However it is doubtful if many people understand the greatness and significance of the achievements of this sad misunderstood genius. Heaviside was : *A mathematical thinker whose work long failed to secure the recognition its brilliance deserved ...*

Test-Exercises

T1.1. Let f, g and h be the functions from $\mathbb{R}$ into itself defined by $f(x) := 1/(1+x^2)$, $g(x) := |x|$ and $h(x) := x+1$. Write the formulas for the compositions fg, fh, gh, gf, hg, hf and examine which of these functions are equal.

T1.2. **a).** For which $a, b, c \in \mathbb{R}$, the function $f:\mathbb{R} \to \mathbb{R}$ with $f(x) := ax^2 + bx + c$ is bijective?

b). Let $a, b, c, d \in \mathbb{R}$ and let $f:\mathbb{R} \to \mathbb{R}$, $g:\mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) := ax+b$, $g(x) := cx+d$. Give necessary and sufficient conditions for the equality $f \circ g = g \circ f$.

T1.3. Which of the following maps from $\mathbb{R} \times \mathbb{R}$ into itself (where is each case the value for $(x, y) \in \mathbb{R} \times \mathbb{R}$ is given) are injective resp. surjective resp. bijective. In the bijective case give the inverse map.

$$(y, 3)\,; \quad (x + y^2, y + 2)\,; \quad (xy, x + 1)\,; \quad (xy, x + y)\,; \quad (2x^2 - y, x + y)\,;$$
$$(x - y, x^2 - y^2)\,; \quad (xy, x^2 - y^2)\,; \quad \left(x\big/\sqrt{1 + x^2 + y^2}\,,\ y\big/\sqrt{1 + x^2 + y^2}\,\right).$$

Is the map $\mathbb{R} \times \mathbb{R} \setminus \{(0,0)\} \to \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$ $(x, y) \mapsto \left(x/(x^2+y^2)\,,\, y/(x^2+y^2)\right)$ injective, resp. surjective resp. bijective? In the bijective case give the inverse map.

T1.4. Let $f : X \to Y$ be a map.

a). The following statements are equivalent:

(i) f is injective.

(ii) For all sets Z and all maps $g_1 : Z \to X$ and $g_2 : Z \to X$, the equality $fg_1 = fg_2$ implies $g_1 = g_2$.

Moreover, if $X \neq \emptyset$, then these statements are further equivalent to:

(iii) There exists a map $g : Y \to X$ such that $gf = \mathrm{id}_X$.

b). The following statements are equivalent:

(i) f is surjective.

(ii) For all sets W and all maps $h_1 : Y \to W$ and $h_2 : Y \to W$, the equality $h_1 f = h_2 f$ implies $h_1 = h_2$.

(iii) There exists a map $h : Y \to X$ such that $fh = \mathrm{id}_Y$. (Such a map h is called a *section* to f.)