# MA-231 Topology

## 4. Metric Spaces

August 27, 2004 ; Submit solutions **before 10:00 AM ; September 06, 2004.**

**Maurice René Fréchet**† **(1878-1973)**

**Felix Hausdorff**†† **(1868-1946)**

**4.1.** Let $X$ be any set and let $d : X \times X \to \mathbb{R}$ be a function.

**a).** Suppose that: $d(x,x) = 0$ for all $x \in X$, $d(x,y) \neq 0$ for all $x, y \in X, x \neq y$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. Prove that $d$ is a metric on $X$. ( Note that $d(x,y) \geq 0$ and $d(x,y) = d(y,x)$ are not being assumed; they must be proved!)

**b).** Suppose that $d$ satisfy the axioms for a metric except that the triangle inequality is reversed, i.e. $d(x,y) \geq d(y,z) + d(z,x)$ for all $x, y, z \in X$. Prove that $X$ has at most one point.

**4.2.** Let $X = \mathbb{Q}$ be the set of rational numbers and let $c$ be a fixed real number with $c \in (0,1)$. Further, let $p$ be a fixed prime number. Every non-zero rational number $x$ can be written in the form $x = p^\alpha a/b$, where $a, b \in \mathbb{Z}$, $p \not| a$ [1]) and $p \not| b$ and where $\alpha \in \mathbb{Z}$. We put $v_p(x) := \alpha$ and $v_p(0) = \infty$. Define $|x|_p = c^{v_p(x)}$ and put $|0|_p := 0$. The map $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ defined by $x \mapsto v_p(x)$ is called the $p$-adic valuation of $\mathbb{Q}$.

**a).** The function $d_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by $d_p(x,y) := |x-y|_p$ is a metric on $\mathbb{Q}$. Moreover, $d_p$ satisfies the ultrametric inequality, see T3.1.(**Hint:** First check that (i) $|x|_p \geq 0$ and equals to 0 if and only if $x = 0$. (ii) $|xy|_p = |x|_p |y|_p$ for all $x, y \in \mathbb{Q}$. (iii) $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$.)

**b).** Is the sequence $(p^n), n \in \mathbb{N}$ convergent in the metric space $(\mathbb{Q}, d_p)$ ?

**4.3.** Let $(X, d)$ be a metric space. Show that

**a).** Any subset of $X$ is an intersection of open subsets in $X$.(**Hint:** $X \setminus \{x\}$ is open in $X$ for every $x \in X$.)

**b).** If every intersection of open subsets in $X$ is an open subset in $X$, then $X$ is discrete.

**c).** If $X$ is infinite, then there exists an open subset $U$ in $X$ such that both $U$ and $X \setminus U$ are infinite. (**Hint:** We may assume that $X$ has only finitely many isolated points. Now, use T4.4-c).)

**4.4.** Let $(X, d)$ be a metric space.

**a).** Prove that the following statements are equivalent:

(i) Every subset of $X$ is either open or closed in $X$. (ii) At most one point of $X$ is not isolated.

**b).** If $X$ is infinite, then there exists an infinite subset of $X$ which is discrete as a metric space in its own right. — For example, the set of integers $\mathbb{Z}$ is a discrete subspace of the real line $\mathbb{R}$. (**Hint:** Use Zorn's Lemma.)

**4.5.** Let $(X, d)$ be a metric space.

**a).** If there are only countably many open subsets in $X$, then prove that $X$ is countable.

**b).** Prove that disjoint closed subsets in $X$ can be enlarged to disjoint open subsets in $X$.

**c).** The closure of a countable subset has the cardinality at most $\mathfrak{c}$ (= the cardinality of the continuum).

---

On the other side one can see (simple) test-exercises; their solutions need not be submitted.

[1]) means " $p$ does not divide $a$ ".

## Test-Exercises

**T4.1.** Let $(X, d)$ be a metric space. We say that $d$ satisfies the u l t r a m e t i c i n e q u a l i t y if $d(x, z) \leq \text{Max}\,\{d(x, y)\,,\, d(y, z)\}$ for all $x, y, z \in X$. For example, the trivial metric satisfies the untrametric inequality. Is the Euclidean space $(\mathbb{R}^n, d)$ ultrametric ? The untrametric inequality is stronger than the triangle inequality.

Let $(X, d)$ be a metric space which satisfy the ultrametric inequality.

**a).** If $d(x, y) \neq d(y, z)$ for $x, y, z \in X$, then $d(x, z) = \text{Max}\,\{d(x, y)\,,\, d(y, z)\}$. —( **Remark :** This result has the following geometric interpretation : every triangle in such a metric space is isosceles and its base has length less than or equal to that of the equal sides.)

**b).** Show that every point of a ball $\text{B}(x; r)$ is a center, i.e. $\text{B}(x; r) = \text{B}(z; r)$ for every $z \in \text{B}(x; r)$.

**c).** If $B$ and $B'$ are two open balls in $X$ with $B \cap B' \neq \emptyset$, then show that one of them contains other, i.e. either $B \subseteq B'$ or $B' \subseteq B$.

**T4.2.** Let $(X, d)$ be a metric space.

**a).** For $x \in X$ and positive real numbers $r, s$ with $r < s$, the subset $\text{A}(x; r, s) : \{y \in X \mid r < d(x, y) < s\}$ is an oepn subset in $X$. — (A reasonable name for such a set is "*open ring*" or *open annulus*")

**b).** Prove that a closed ball in a $X$ is a closed subset in $X$.

**c).** For $x \in X$ and a non-negative real number $r$, the subset $\text{S}(x; r) := \{y \in X \mid d(x, y) = r\}$ is called the s p h e r e with radius $r$ and the center $x$. Prove that a sphere is a closed subset in $X$.

**d).** For $x \in X$ and positive real numbers $r, s$ with $r < s$, the subset $\overline{\text{A}}(x; r, s) : \{y \in X \mid r \leq d(x, y) \leq s\}$ is a closed subset in $X$. — (This is called a "*closed ring*" or *closed annulus*")

**T4.3. a).** Prove that any bounded open subset of the $\mathbb{R}$ is a countable union of disjoint open intervals.

**b).** Prove that any non-empty open subset of $\mathbb{R}$ has cardinality $\mathfrak{c} := \text{card}(\mathbb{R})$ — the cardinal number of the c o n t i n u u m .

**T4.4. a).** In the Euclidean space $(\mathbb{R}^n, d), n \geq 1$, there are no isolated points.

**b).** Show that every point of a finite open subset of a metric space $(X, d)$ is an isolated point in $X$.

**c).** Let $x$ be a point in a metric space $(X, d)$. Prove that the following statements are equivalent :

(i) $x$ is not isolated. (ii) Every neighbourhood of $x$ contains an infinite number of points of $X$.

**T4.5.** Let $(X, d)$ be a metric space.

**a).** Prove that the following statements are equivalent :

(i) $X$ is discrete. (ii) Every convergent sequence is ultimately constant. (iii) $X$ has no limit points. (iv) The closure of every open subset is open.

**b).** Let $x, y \in X$ be two distinct points. Prove that there exist open neighbourhoods $U$ and $V$ of $x$ and $y$ such that $\overline{U} \cap \overline{V} = \emptyset$.

**c).** Let $Y$ be a subset of $X$ and let $x \in X$. Prove that $x$ is a limit point of $Y$ if and only if there exists a sequence $y_n, n \in \mathbb{N}$ in $Y \setminus \{x\}$ converging to $x$. Further, prove that the $y_n$'s can be chosen so that the real numbers $d(y_n, x)$ decrease strictly monotonically.

---

† **M a u r i c e R e n é F r é c h e t ( 1 8 7 8 - 1 9 7 3 )** Maurice René Fréchet was born on 2 Sept 1878 in Maligny, Yonne, Bourgogne, France and died on 4 June 1973 in Paris, France. Maurice Fréchet was a student of Hadamard 's and, under his supervision, Fréchet wrote an outstanding dissertation in 1906, introducing the concept of a metric space. He did not invent the name "*metric space*" which is due to Hausdorff.

A versatile mathematician, Fréchet served as professor of mechanics at the University of Poitiers (1910-19) and professor of higher calculus at the University of Strasbourg (1920-27). He held several different positions in the field of mathematics at the University of Paris (1928-48) including lecturer of the calculus of probabilities, professor of differential and integral calculus and professor of the calculus of probabilities.

Fréchet made major contributions to the topology of point sets and defined and founded the theory of abstract spaces. Fréchet also made important contributions to statistics, probability and calculus. In his dissertation of 1906, mentioned above, he investigated functionals on a metric space and formulated the abstract notion of compactness. In 1907 he discovered an integral representation theorem for functionals on the space of quadratic Lebesgue integrable functions. A similar result was discovered independently by Riesz.

†† **F e l i x H a u s d o r f f ( 1 8 6 8 - 1 9 4 2 )** Felix Hausdorff was born on 8 Nov 1868 in Breslau, Germany (now Wroclaw, Poland) and died on 26 Jan 1942 in Bonn, Germany. Felix Hausdorff graduated from Leipzig in 1891 and then taught there until 1910 when he went to Bonn. Within a year of his appointment to Leipzig he was offered a post at Göttingen but, rather surprisingly given Göttingen's reputation, he turned it down. Hausdorff worked at Bonn until 1935 when he was forced to retire by the Nazi regime. Although as early as 1932 he sensed the oncoming calamity of Nazism he made no attempt to emigrate while it was still possible. As a Jew his position became more and more difficult. In 1941 he was scheduled to go to an internment camp but managed to avoid being sent. However by 1942 he could no longer avoid being sent to the internment camp and, together with his wife and his wife's sister, he committed suicide.

Hausdorff's main work was in topology and set theory. He introduced the concept of a partially ordered set and from 1906 to 1909 he proved a series of results on ordered sets. In 1907 he introduced special types of ordinals in an attempt to prove Cantor's continuum hypothesis. He also posed a generalisation of the continuum hypothesis by asking if 2 to the power $\aleph_n$ was equal to $\aleph_{n+1}$. Hausdorff proved further results on the cardinality of Borel sets in 1916.

Building on work by Fréchet and others, he created a theory of topological and metric spaces with Grundzüge der Mengenlehre (1914). Earlier results on topology fitted naturally into the framework set up by Hausdorff. In 1919 he introduced the notion of Hausdorff dimension, which was a real number lying between the topological dimension of an object and 3. It is used to study objects such as Koch's curve. He also introduced the *Hausdorff measure* and the term "*metric space*" is due to him.