MA-231 Topology

6. Continuous Maps

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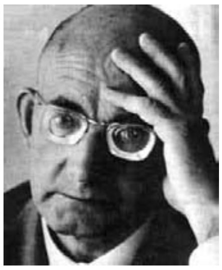
Robert Lee Moore†
(1882-1974)



Pavel Sergeevich Aleksandrov††
(1896-1982)

6.1. Let X and Y be topological spaces.

a). Show that all open (respectively bounded closed) intervals in $\mathbb{R}$ are homeomorphic.

b). Show that the property that every real-valued continuous functions on X assumes its maximum is a topological property (see 6.2 below). In particular, the closed unit interval $[0, 1]$ is not homeomorphic to $\mathbb{R}$.

c). Show that there is no analog of the Cantor-Berstein theorem for topological spaces. (Recall that the Cantor-Berstein theorem states that: if X, Y are sets and if $f : X \to Y$, $g : Y \to X$ are injective maps, then there exists a bijective map $h : X \to Y$. The analog for topological spaces would be : *Whenever X can be embedded*[1]) *in Y and Y can be embedded in X, then X and Y are homeomorphic.*) Find a counter example.(**Hint:** Use b) above.)

6.2. (Topological properties) A topological property is a property of a topological spaces which, if posed by all spaces homeomorphic to X, is possessed by all spaces homeomorphic to X.

a). Show that each of the following expresses a topological property of a topological space X :

(1) X has cardinal number $\aleph$, (2) the topology of X has cardinal number $\aleph$, (3) the topology on X has a base whose cardinal number is $\aleph$. (4) there is a subset of X with cardinal number $\aleph$ whose closure is X, (5) X is metrizable (resp. first countable, second countable, separable).

b). Show that each of the following expresses a property of X which is not a topological property :

(1) the topology of X is generated by the metric d, (2) X is a subset of $\mathbb{R}$.

6.3. Let X be a topological space and let $C_{\mathbb{R}}(X)$ (respectively, $C^*_{\mathbb{R}}(X)$) denote the set of all continuous (respectively, continuous bounded) real-valued functions on X. We can define addition, multiplication and scalar multiplication of functions in $C_{\mathbb{R}}(X)$ pointwise :

$(f+g)(x) := f(x)+g(x)$, $(f \cdot g)(x) := f(x) \cdot g(x)$, $(a \cdot f)(x) := a \cdot f(x)$, for $a \in \mathbb{R}$ and $x \in X$.

a). On $C_{\mathbb{R}}(X)$ the operations of addition, multiplication and scalar multiplication given above gives a structure of an algebra over $\mathbb{R}$.

b). On $C^*_{\mathbb{R}}(X)$ define $\|f\| : \sup_{x\in X} |f(x)|$. Then the operations of addition, scalar multiplication and $\|\cdot\|$ gives a structure of a normed linear space over $\mathbb{R}$.

c). On $C_{\mathbb{R}}(X)$ define the order by $f \leq g$ if and only if $f(x) \leq g(x)$ for each $x \in X$. With this order both $C_{\mathbb{R}}(X)$ and $C^*_{\mathbb{R}}(X)$ are lattices. (**Hint:** For $f, g \in C_{\mathbb{R}}(X)$ (respectively, $C^*_{\mathbb{R}}(X)$), both $m(x) := \min\{f(x), g(x)\}$ and $M(x) := \max\{f(x), g(x)\}$ belong to $C_{\mathbb{R}}(X)$ (respectively, $C^*_{\mathbb{R}}(X)$).)

(**Remark:** Study of the interaction between the algebraic and lattice properties of $C_{\mathbb{R}}(X)$ and $C^*_{\mathbb{R}}(X)$ and topological properties of X is still actively being carried on. Some questions of importance in this direction are : (i) For what class of spaces is it true that X and Y are homeomorphic if and only if $C^*_{\mathbb{R}}(X)$ and $C^*_{\mathbb{R}}(Y)$ (or $C_{\mathbb{R}}(X)$

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[1]) We say that a map $f : X \to Y$ is an embedding of X into Y if f is injective, continuous and the inverse $f^{-1} : f(X) \to X$ is also continuous and say that X is embedded in Y by f. Therefore, X is embeded into Y by $f : X \to Y$ if and only if f is a homeomorphism between X and some subspace of Y.

and $C_{\mathbb{R}}(Y)$) are isomorphic ($\mathbb{R}$- algebras). (ii) How are topological properties of X refelcted in algebraic and lattice properties of $C^*_{\mathbb{R}}(X)$ and $C_{\mathbb{R}}(X)$. (iii) What properties of a ring R (usually with a lattice structure) will ensure that R is isomorphic to $C_{\mathbb{R}}(X)$ for some topological space X.)

6.4. (The group of homeomorphisms) For any topological space X, let $H(X)$ denote the group of homeomorphisms of X onto itself, with composition as the group operation. Let X and Y be two topological spaces.

a). If $f : X \to Y$ is a homeomorphism, then the map $H(f) : H(X) \to H(Y)$ defined by $H(f)(\varphi) = f \circ \varphi \circ f^{-1}$ is an isomorphism of groups.

b). A central and obvious questions is: If $\Phi : H(X) \to H(Y)$ is an isomorphism of groups, is there a homeomorphism $f : X \to Y$ such that $H(f) = \Phi$. (**Hint:** Let $X = [0, 1]$ and $Y = (0, 1)$. Then the restriction map $\Phi : H(X) \to H(Y), h \mapsto h|Y$ is an isomorphism of groups, but there is no homeomorphism of X onto Y (why?). — Affirmative answers are available, however, for suitably restricted classes of spaces.)

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† **Robert Lee Moore (1882-1974)** Robert Lee Moore was born in 14 Nov 1882 in Dallas, Texas, USA and died on 4 Oct 1974 in Austin, Texas, USA. Robert Lee Moore's father, Charles Jonathan Moore, owned a hardware store in Dallas. Originally from Connecticut, Charles had moved to the south of the United States during the civil war to fight on the side of the South. Robert Lee Moore's mother was Louisa Ann Moore and she did not need to change her name on marrying Charles since her maiden name was also Moore. Charles and Louisa had six children, with Robert being the second youngest in the family. Robert received a good education at a private high school in Dallas, and before he entered university he had learnt university level calculus by studying the university textbooks.

He entered the University of Texas in 1898 and there he took courses by Halsted and Dickson . He graduated with a Sc.B. in 1901 and after a year as a teaching fellow at the University of Texas, Moore spent the academic year 1902-03 as a mathematics instructor at the High School in Marshall, Texas. In fact Moore would have remained at Texas University rather than spend the year teaching in a high school but, for some reason which is not clear, the university regents refused to renew his appointment despite strong protests from Halsted.

Now Halsted had suggested a problem in one of his classes which had led Moore to prove that one of Hilbert's geometry axioms was redundant. Eliakim Moore , who was the head of mathematics at Chicago University, heard of this contribution and, since his research interests at the time were precisely on the foundations of geometry, Eliakim Moore organised the award of a scholarship that would allow Robert Moore to study for his doctorate in Chicago. We should note that despite the fact that Eliakim Moore and Robert Moore shared the same surname and the same research interests, they were not related. Veblen supervised Moore's Ph.D. at University of Chicago and the degree was awarded in 1905 for a dissertation entitled Sets of Metrical Hypotheses for Geometry.

It was while Moore was attending lectures in Chicago during this period that he first hit on his original teaching methods. With his quick mind and restless spirit he found the lecture method rather boring - in fact, mind dulling. To liven up a lecture he would run a race with his professor by seeing if he could discover the proof of an announced theorem before the lecturer had finished his presentation. Quite frequently he won the race. But in any case, he felt that he was better off from having made the attempt.

Moore spent the year 1905-06 as an assistant professor at the University of Tennessee, then two years as an instructor at Princeton University. In 1908 he was appointed as an instructor at the Northwestern University and then, after three years there, he went to the University of Pennsylvania in 1911. The year before, in 1910, he had married Margaret MacLelland Key of Brenham, Texas; they had no children. After a promotion to assistant professor at the University of Pennsylvania in 1916, he remained there for a further four years.

It was at the University of Pennsylvania that Moore first tried out his teaching methods in a Foundations of Geometry course he taught there. He began to have success with what became known as the Moore Method of teaching. Here was a fresh, relatively new area where Moore had himself tested the difficulty of some of the theorems. We shall describe the Moore Method below.

Moore was appointed to the staff at the University of Texas in 1920 as an associate professor, being made a full professor three years later. Moore was delighted to return to the University of Texas, his home university. By the time he was appointed in 1920 he had published 17 papers on point-set topology (a term which he coined). For his doctoral thesis Moore had worked on the foundations of topology. In 1915 he published On a set of postulates which suffice to define a number-plane published in the Transactions of the American Mathematical Society. Writing about this paper in 1927, Chittenden wrote:-

The importance of the regularly and perfectly separable, therefore metric, spaces in the analysis of continua is indicated by the fact that nine years before the publication of the discoveries of Urysohn, R L Moore assumed these properties in the first of a system of axioms for the foundations of plane analysis situs.

Moore wrote up his work on point-set topology in the important book Foundations of point set topology published in 1932. This volume, published in the Colloquium Lectures Series of the American Mathematical Society , arose from the colloquium lectures which Moore gave in 1929 and is a self-contained introduction to the topic concentrating on Moore's own contributions to the subject.

We should comment on Moore's teaching methods, for their success influenced others to use similar methods. These methods are described by F Burton Jones, who himself was a student of Moore, and himself taught very successfully with a modified version.

Moore would begin his graduate course in topology by carefully selecting the members of the class. If a student had already studied topology elsewhere or had read too much, he would exclude him (in some cases, he would run a separate class for such students). The idea was to have a class as homogeneously ignorant (topologically) as possible. He would usually caution the group not to read topology but simply to use their own ability. Plainly he wanted the competition to be as fair as possible, for competition was one of the driving forces. ...

Having selected the class he would tell them briefly his view of the axiomatic method: there were certain undefined terms (e.g., "point" and "region") which had meaning restricted (or controlled) by the axioms (e.g., a region is a point set). He would then state the axioms that the class was to start with ...

After stating the axioms and giving motivating examples to illustrate their meaning he would then state some definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened,' omitted, or partially omitted.

When the class returned for the next meeting he would call on some student to prove Theorem 1 . After he became familiar with the abilities of the class members, he would call on them in reverse order and in this way give the more unsuccessful students first chance when they did get a proof. He was not inflexible in this procedure but it was clear that he preferred it.

When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present his proof. Then the other students, especially those who had not been able to discover a proof, would make sure that the proof presented was correct and convincing. Moore sternly prevented heckling. This was seldom necessary because the whole atmosphere was one of a serious community effort to understand the argument.

When a flaw appeared in a "proof" everyone would patiently wait for the student at the board to "patch it up." If he could not, he would sit down. Moore would then ask the next student to try or if he thought the difficulty encountered was sufficiently interesting, he would save that theorem until next time and go on to the next unproved theorem (starting again at the bottom of the class). Mary Ellen Rudin, who was also a student of Moore's presents a similar picture: His way of teaching was to present you with things that had not yet been proved, and with all kinds of things which might turn out to have a counterexample, and sometimes unsolved problems - that is, unsolved by anyone, not only unsolved by you. So you had some idea of what it meant to be a mathematician - more than the average undergraduate does today.

Although the Moore Method proved good for Mary Ellen Rudin, she understood that it was not right for everyone:-

I wouldn't for anything have let my children go to school with Moore! That is, I think that he was destructive to anyone who didn't fit exactly into his pattern, he did not succeed in giving the people that worked with him an education. It's a mistake to go to school under those circumstances in general.

Moore taught at Texas until he was 86 years old, and he wished to carry on teaching but the University authorities forced him to retire. A number of students strongly supported his bid to remain in post but to no avail. The university authorities were not concerned at his abilities to teach, rather it was the great success of his methods which made his employers fear that bright young mathematicians might not wish to teach there due to his continuing dominating influence. In the picture above he is aged 87 and still in his office in Austin, Texas. The University of Texas did Moore a great honour, however, for in 1973 they named a new physics, mathematics and astronomy building after him.

A strong supporter of the American Mathematical Society , Moore was an editor of the Colloquium Publications from 1929 to 1936, being editor-in-chief from 1930 to 1933. He was president of the American Mathematical Society from 1936 to 1938. He was elected to the National Academy of Sciences in 1931.

Finally we should make some negative comments about his bigoted attitudes. The quotation below is from a personal communication from Chandler Davis which is based on:- ... conversations and correspondence with my good friend E E Moise.

Chandler Davis writes: R L Moore was firmly anti-black, refusing to teach any black students. He was pretty bigoted against women and Jews too, as many anecdotes attest. Two of his supervisees who went on to brilliant careers and who remained grateful for his teaching were, however, Mary Ellen Rudin and E E Moise. Moore took quite some time, I am told, to adjust to working with a woman and with a Jew, but after he got used to it he treated them well. (Moise was of mixed background, but as he bore the name of his Jewish grandfather he was a Jew in Moore's eyes.) As Chandler Davis suggests, Mary Ellen Rudin was certainly happy with Moore. He encouraged people to believe in themselves as mathematicians because he felt that this was one of the principal tools for doing mathematics - to have confidence. ... I probably would not be a mathematician had I not worked with Moore.

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†† **Pavel Sergeevich Aleksandrov (1896-1982)** Pavel Sergeevich Aleksandrov was born on 7 May 1896 in Bogorodsk (also called Noginsk), Russia and died on 16 Nov 1982 in Moscow, USSR.

Like most Russian mathematicians there are different ways to transliterate Aleksandrov's name into the Roman alphabet. The most common way, other than Aleksandrov, is to write it as Alexandroff.

Pavel Sergeevich Aleksandrov's father Sergej Aleksandrovich Aleksandrov was a medical graduate from Moscow University who had decided not to follow an academic career but instead had chosen to use his skills in helping people and so he worked as a general practitioner in Yaroslavskii. Later he worked in more senior positions in a hospital in Bogorodskii, which is where he was when Pavel Sergeevich was born.

When Pavel Sergeevich was one year old his father moved to Smolensk State hospital, where he was to earn the reputation of being a very fine surgeon, and the family lived from this time in Smolensk. The city of Smolensk is on the Dnieper River 420 km west of Moscow. Pavel Sergeevich's early education was from his mother, Tsezariya Akimovna Aleksandrova, who applied all her considerable talents to bringing up and educating her children. It was from her that Aleksandrov learnt French and also German. His home was one that was always filled with music as his brothers and sisters all had great talent in that area.

The fine start which his mother gave him meant that he always excelled at the grammar school in Smolensk which he attended. His mathematics teacher Alexsander Romanovich Eiges soon realised that his pupil had a remarkable talent for the subject and :

... at grammar school he studied celestial mechanics and mathematical analysis. But his interest was mainly directed towards fundamental problems of mathematics: the foundations of geometry and non-euclidean geometry. Eiges had a proper appreciation of his pupil and exerted a decisive influence on his choice of a career in mathematics.

In 1913 Aleksandrov graduated from the grammar school being dux of the school and winning the gold medal. Certainly at this time he had already decided on a career in mathematics, but he had not set his sights as high as a university teacher, rather he was aiming to become a secondary school teacher of mathematics. Eiges was the role model who he was aspiring to match at this stage, for Eiges had done more than teach Aleksandrov mathematics, he had also influenced his tastes in literature and the arts.

Aleksandrov entered Moscow University in 1913 and immediately he was helped by Stepanov. Stepanov , who was working at Moscow University, was seven years older than Aleksandrov but his home was also in Smolensk and he often visited the Aleksandrov home there. Stepanov was an important influence on Aleksandrov at this time and suggested that Aleksandrov join Egorov's seminar even in the first year of his studies in Moscow. In Aleksandrov's second year of study he came in contact with Luzin who had just returned to Moscow. Aleksandrov wrote :

After Luzin's lecture I turned to him for advice on how best to continue my mathematical studies and was struck most of all by Luzin's kindness to the man addressing him - an 18-year old student ... I then became a student of Luzin, during his most creative period ... To see Luzin in those years was to see a display of what is called an inspired relationship to science. I learnt not only mathematics from him, I received also a lesson in what makes a true scholar and what a university professor can and should be. Then, too, I saw that the pursuit of science and the raining of young people in it are two facets of one and the same activity - that of a scholar.

Aleksandrov proved his first important result in 1915, namely that every non-denumerable Borel set contains a perfect subset. It was not only the result which was important for set theory, but also the methods which Aleksandrov used which turned out to be one of the most useful methods in descriptive set theory. After Aleksandrov's great successes Luzin did what many a supervisor might do, he realised that he had one of the greatest mathematical talents in Aleksandrov so he thought that it was worth asking him to try to solve the biggest open problem in set theory, namely the continuum hypothesis.

After Aleksandrov failed to solve the continuum hypothesis (which is not surprising since it can neither be proved or disproved as was shown by Cohen in the 1960s) he thought he was not capable of a mathematical career. Aleksandrov went to Novgorod-Severskii and became a theatre producer. He then went to Chernikov where, in addition to theatrical work, he lectured on Russian and foreign languages, becoming friends with poets, artists and musicians. After a short term in jail in 1919 at the time of the Russian revolution, Aleksandrov returned to Moscow in 1920. Luzin and Egorov had built up an impressive research group at the University of Moscow which the students called 'Luzitania' and they, together with Privalov and Stepanov, were very welcoming to Aleksandrov on his return.

It was not an immediate return to Moscow for Aleksandrov, however, for he spent 1920-21 back home in Smolensk where he taught at the University. During this time he worked on his research, going to Moscow about once every month to keep in touch with the mathematicians there and to prepare himself for his examinations. At around this time Aleksandrov became friendly with Urysohn, who was a member of 'Luzitania', and the friendship would soon develop into a major mathematical collaboration.

After taking his examinations in 1921, Aleksandrov was appointed as a lecturer at Moscow university and lectured on a variety of topics including functions of a real variable, topology and Galois theory. In July 1922 Aleksandrov and Urysohn went to spend the summer at Bolshev, near to Moscow, where they began to study concepts in topology. Hausdorff, building on work by Fréchet and others, had created a theory of topological and metric spaces in his famous book Grundzüge der Mengenlehre published in 1914. Aleksandrov and Urysohn now began to push the theory forward with work on countably compact spaces producing results of fundamental importance. The notion of a compact space and a locally compact space is due to them.

In the summers of 1923 and 1924 Aleksandrov and Urysohn visited Göttingen and impressed Emmy Noether, Courant and Hilbert with their results. The mathematicians in Göttingen were particularly impressed with their results on when a topological space is metrisable. In the summer of 1924 they also visited Hausdorff in Bonn and he was fascinated to hear the major new directions that the two were taking in topology. However while visiting Hausdorff in Bonn ([3] and [4]):-

Every day Aleksandrov and Urysohn swam across the Rhine - a feat that was far from being safe and provoked Hausdorff's displeasure.

Aleksandrov and Urysohn then visited Brouwer in Holland and Paris in August 1924 before having a holiday in the fishing village of Bourg de Batz in Brittany. Of course mathematicians continue to do mathematics while on holiday and they were both working hard. On the morning of 17 August Urysohn began to write a new paper but tragically he drowned while swimming in the Atlantic later that day. Aleksandrov determined that no ideas of his great friend and collaborator should be lost and he spent part of 1925 and 1926 in Holland working with Brouwer on preparing Urysohn's paper for publication.

The atmosphere in Göttingen had proved very helpful to Aleksandrov, particularly after the death of Urysohn, and he went there every summer from 1925 until 1932. He became close friends with Hopf and the two held a topological seminar in Göttingen. Of course Aleksandrov also taught in Moscow University and from 1924 he organised a topology seminar there. At Göttingen, Aleksandrov also lectured and participated in Emmy Noether's seminar. In fact Aleksandrov always included Emmy Noether and Hilbert among his teachers, as well as Brouwer in Amsterdam and Luzin and Egorov in Moscow.

From 1926 Aleksandrov and Hopf were close friends working together. They spent some time in 1926 in the south of France with Neugebauer. Then Aleksandrov and Hopf spent the academic year 1927-28 at Princeton in the United States. This was an important year in the development of topology with Aleksandrov and Hopf in Princeton and able to collaborate with Lefschetz, Veblen and Alexander. During their year in Princeton, Aleksandrov and Hopf planned a joint multi-volume work on Topology the first volume of which did not appear until 1935. This was the only one of the three intended volumes to appear since World War II prevented further collaboration on the remaining two volumes. In fact before the joint work with Hopf appeared in print, Aleksandrov had begun yet another important friendship and collaboration.

In 1929 Aleksandrov's friendship with Kolmogorov began and they : *... journeyed a lot along the Volga, the Dnieper, and other rivers, and in the Caucuses, the Crimea, and the south of France.* The year 1929 marks not only the beginning of the friendship with Kolmogorov but also the appointment of Aleksandrov as Professor of Mathematics at Moscow University. In 1935 Aleksandrov went to Yalta with Kolmogorov, then finished the work on his Topology book in the nearby Crimea and the book was published in that year. The 'Komarovski' period also began in that year:

Over the last forty years, many of the events in the history of mathematics in the University of Moscow have been linked with Komarovka, a small village outside Moscow. Here is the house owned since 1935 by Aleksandrov and Kolmogorov. Many famous foreign mathematicians also visited Komarovka - Hadamard, Fréchet, Banach, Hopf, Kuratowski, and others.

In 1938-1939 a number of leading mathematicians from the Moscow University, among them Aleksandrov, joined the Steklov Mathematical Institute of the USSR Academy of Sciences but at the same time they kept their positions at the University.

Aleksandrov wrote about 300 scientific works in his long career. As early as 1924 he introduced the concept of a locally finite covering which he used as a basis for his criteria for the metrisability of topological spaces. He laid the foundations of homology theory in a series of fundamental papers between 1925 and 1929. His methods allowed arguments of combinatorial and algebraic topology to be applied to point set topology and brought together these areas. Aleksandrov's work on homology moved forward with his homological theory of dimension around 1928-30

Aleksandrov was the first to use the phrase 'kernel of a homomorphism' and around 1940-41 he discovered the ingredients of an exact sequence. He worked on the theory of continuous mappings of topological spaces. In 1954 he organised a seminar on this last topic aimed at first year students at Moscow University and in this he showed one of the aspects of his career which was of major importance to him, namely the education of students. This is described as:

To the training of these students and those who came after them, Aleksandrov literally devoted all his strength. His influence on the class of young men studying topology under him was never purely mathematical, however real and significant that was. There were physical days exercise on topological walks, in long outings lasting several days by boat, ... in swimming across the Volga or other broad stretches of water, in skiing excursions lasting for hours on the slopes outside Moscow, slopes to which Aleksandrov gave striking, fantastic names...

Many honours were given to Aleksandrov for his outstanding contribution to mathematics. He was president of the Moscow Mathematical Society from 1932 to 64, vice president of the International Congress of Mathematicians from 1958 to 62, a corresponding member of the USSR Academy of Sciences from 1929 and a full member from 1953. Many other societies elected Aleksandrov to membership including the Göttingen Academy of Sciences, the Austrian Academy of Sciences, the Leopoldina Academy in Halle, the Polish Academy of Sciences, the National Academy of Sciences of the United States, the London Mathematical Society, the American Philosophical Society, and the Dutch Mathematical Society.

He edited several mathematical journals, in particular the famous Soviet Journal Uspekhi Matematicheskikh Nauk, and he received many Soviet awards, including the Stalin Prize in 1943 and five Orders of Lenin.

Today the Department of General Topology and Geometry of Moscow State University is Russia's leading centre of research in set-theoretic topology. After Aleksandrov's death in November 1982, his colleagues from the Department of Higher Geometry and Topology, in which he had held the chair, sent a letter to Moscow University's rector A A Logunov proposing that one of Aleksandrov's former students should become Head of the Department, to preserve Aleksandrov's scientific school. On 28 December 1982 the rector issued a circular creating the Department of general topology and Geometry. Vitaly Vitalievich Fedorchuk was elected Head of the Department.

Also in memory of Aleksandrov's contributions to topology at Moscow University and his work with the Moscow Mathematical Society, there is an annual topological symposium Aleksandrov Proceedings held every May.