MA-231 Topology

7. Product Spaces

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Andrei Nikolaevich Tikhonov† **(1906-1993)**

Heinrich Franz Friedrich Tietze†† **(1880-1964)**

7.1. (Closure, Interior and Boundary in products spaces) Let X and Y be topological spaces and let $X \times Y$ be the product space. Let A and B be subsets of X and Y respectively. Then

a). $(A \times B)^\circ = A^\circ \times B^\circ$.

b). $\overline{(A \times B)} = \overline{A} \times \overline{B}$. —(**Remark:** the part b) can be extended to infinite products, while the part a) can be extended only to finite products.)

c). $\partial(A \times B) = \big(\overline{A} \times \partial(B)\big) \cup \big((\partial(A) \times \overline{B}\big)$.

d). Let X_i, $i \in I$ be a family of non-empty topological spaces and let $A_i \subseteq X_i$, $i \in I$. Then $\prod_{i\in I} A_i$ is dense in the product space $\prod_{i\in I} X_i$ iif and only if A_i is dense in X_i for each $i \in I$.

7.2. Let X_i, $i \in I$ be a family of non-empty topological spaces and let $X = \prod_{i\in I} X_i$ be the product space.

a). If V is an open set in X, then $p_i(V) = X_i$ for almost all $i \in I$.

b). Let $a_i \in X_i, i \in I$ be a fixed point. Then the subset $X'_{i_0} := \{x \in X \mid x_i = a_i \text{ whenever } i \in I, i \neq i_0\}$ is homeomorphic to X_{i_0}.

c). Let $a_i \in X_i$, $i \in I$ be a fixed point. Then the subset $Y := \{x \in X \mid x_i = a_i \text{ for almost all } i \in I\}$ is a dense subset of X, i.e. $\mathrm{CL}_X(Y) = X$.

7.3. a). (Associativity of products) Let I be an indexed set and let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a partition of the set I (into disjoint subsets whose union is I) and let X_i, $i \in I$ be a family of topological spaces. The product space $\prod_{\lambda\in\Lambda} \big(\prod_{i\in I_\lambda} X_i\big)$ is homeomorphic to the product space $\prod_{i\in I} X_i$.

(**Hint:** For each $\lambda \in \Lambda$, let $q_\lambda : \prod_{i\in I} X_i \to \prod_{i\in I_\lambda} X_i$ be the map defined by $(x_i)_{i\in I} \mapsto \big((x_i)_{i\in I_\lambda}\big)_{\lambda\in\Lambda}$. Then each q_λ is surjective (axiom of choice!) and for each $j \in I_\lambda$, the map $p_j \circ q_\lambda$ is the projection of $\prod_{i\in I} X_i$ onto its j-th factor and so each q_λ is also continuous. Now check that the map $q : \prod_{i\in I} X_i \to \prod_{\lambda\in\Lambda} \big(\prod_{i\in I_\lambda} X_i\big)$ defined by $x \mapsto (q_\lambda(x))_{\lambda\in\Lambda}$ is bijective, continuous and open and hence a homeomorphism.)

b). Let I be an indexed set and let X_i, Y_i, $i \in I$ be two families of topological spaces. Let $f_i : X_i \to Y_i$ be a family of maps and let $\prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ be the map defined by $(x_i)_{i\in I} \mapsto (f_i(x_i))_{i\in I}$. Then: (1) If each f_i is continuous, so also is $\prod_{i\in I} f_i$. (2) If each f_i is an open map and f_i is surjective for almost all $i \in I$, then $\prod_{i\in I} f_i$ is also an open map.

c). (Commutativity of products) Let I, J be two indexed sets and let X_i, $i \in I$ and Y_j, $j \in J$ be two families of topological spaces. Let $\varphi : I \to J$ be a bijective map. If for each $i \in I$, the topologocal spaces X_i and $Y_{\varphi(i)}$ are homeomorphic, then the product spaces $\prod_{i\in I} X_i$ and $\prod_{j\in J} Y_j$ are homeomorphic. In particular, $\prod_{i\in I} X_i$ is unrestrictedly commutative.

d). Let X be a fixed topological space, I be any infinite indexed set and let $X_i = X$ for each $i \in I$. let $Y := \prod_{i\in I} X_i$ be the product space. Then each product space $\prod_{j\in J} Y$ (consisting of $|J|$ factors Y) with $|J| \leq |I|$ is homeomorphic to Y. (**Hint:** Since $|J| \leq |I|$ and $|I$ is inifinite, we have $|J| \cdot |I| = |I|$. Now apply the part c).)

*__7.4.__[1]) Let C denote the *Cantor set* [2]) with the subspace topology from the usual topology of $\mathbb{R}$. For each $n \in \mathbb{N}^*$, let X_n be the discrete space $\{0, 2\}$.

a). The map $\varphi : \prod_{n\in\mathbb{N}^*} X_n \to \mathrm{C}$ defined by $(a_n)_{n\in\mathbb{N}^*} \mapsto \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, is a homeomorphism of the product space $\prod_{n\in\mathbb{N}^*} X_n$ onto C. — (**Remark :** This part give s an alternative description of the Cantor set : each $x \in \mathrm{I}$ has an expression $(x_1, x_2, \ldots,)$ in ternary form, i.e. each x_i is 0, 1, or 2, obtained by writting $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$. These expressions are unique, except that any number but 1 expressible in ternary expression ending in a sequence of 2's can be re-expressed in an expansion ending in a sequence of 0's (for example, $\frac{1}{3}$ can be written as $(1, 0, 0, \ldots)$ or as $(0, 2, 2, 2, \ldots)$). Then *the Cantor set* C *is precisely the set of points of* I *having ternary expansion without* 1*'s*. For this reason, C is sometimes refered as the Cantor's ternary set. Later, we will see that the product of countably many non-trivial finite discrete spaces is homeomorphic to the Cantor set. For this reason, (possibly uncountable) products of finite discrete spaces are called Cantor spaces. The Cantor spaces occupy a special place in topology. Compactness and discreteness are, in a sense, dual properties and only the cantor spaces carry the banners of both.)

b). For each indexed set I with $|I| \leq \aleph_0$, there exists a homeomorphism $g : \prod_{i\in I} \mathrm{C} \to \mathrm{C}$. (**Hint :** Use the part a) and 7.3-d).)

c). The map $\psi : \prod_{n\in\mathbb{N}^*} X_n \to [0, 1]$ defined by $(a_n)_{n\in\mathbb{N}^*} \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}$, is continuous and surjective.

d). (Peano's Space filling curves [3])) Let I denote the closed unit interval [0, 1]. For $n \in \mathbb{N}$, the n-cube in the Euclidean space is the product space $\mathrm{I}^n := \prod_{i=1}^{n} I$. The (countable) product space $\mathrm{I}^\infty := \prod_{i\in\mathbb{N}} I$ is called the Hilbert cube. For each $k \leq \aleph_0$, there exist surjective continuous maps $f : \mathrm{C} \to I^k$ and $F : I \to I^k$, i.e. a curve going through each point of the k-cube I^k. (**Hint :** Let I be an indexed set with $|I| = k \leq \aleph_0$ and let φ, g, ψ be the maps as in the parts a), b) ,c) respectively. Then the map $f = \left(\prod_{i\in I} \psi\right) \circ \left(\prod_{i\in I} \varphi^{-1}\right) \circ g^{-1} : \mathrm{C} \to \prod_{i\in I} \mathrm{C} \to \prod_{i\in I} \left(\prod_{n\in\mathbb{N}^*} X_n\right) \to \prod_{i\in I} \mathrm{I} = \mathrm{I}^k$ is continuous and surjective. Let p_i, $i \in I$ denote the i-th projection $\prod_{i\in I} \mathrm{I} \to \mathrm{I}$. Extend each $p_i \circ f$ to a continuous map $f_i : \mathrm{I} \to \mathrm{I}$ be defining f_i to be linear on each omitted interval (recall from the remark of the part a) that $\mathrm{C} \subseteq \mathrm{I}$ is obtained by sucessively dropping out middle thirds). Now, the map $F : \mathrm{I} \to \prod_{i\in I} \mathrm{I} = \mathrm{I}^k$ defined by $t \mapsto (f_i(t))_{i\in I}$ is continuous and surjective, since $F|\mathrm{C} = f$ is surjective.)

7.5. Exhibit topological spaces X, Y and Z such that $X \times Y$ is homeomorphic to $X \times Z$, but Y is not homeomorphic to Z. (**Hint :** Let $X = Y = \mathrm{C}$ be the Cantor set and $Z = \{0, 2\}$ be the discrete space. Then Y and Z are not homeomorphic, but both $X \times Y$ and $X \times Z$ are homeomorphic to C by 7.4-b), 7.4-a) and 7.3-c). — **Remark :** It is also true that there are non-homeomorphic topological spaces X and Y such that $X \times X$ and $Y \times Y$ are homeomorphic. Fox, R. H. [4]) have provided example of non-homeomorphic topological spaces X and Y whose squares are homeomorphic.)

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[1]) You should do it if you think you can't, since it will teach you a lot about product spaces!

[2]) **Cantor set** Beginning with the closed unit interval $\mathrm{I} := [0, 1]$ defined subsets $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$ in I as follows : $A_1 := \mathrm{I} \setminus (\frac{1}{3}, \frac{2}{3})$, $A_2 := A_1 \setminus (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$. In general having A_{n-1}, A_n is obtained by removing the open middle thirds from each of the 2^{n-1} closed intervals that make up A_{n-1}. The Cantor set is the subspace $\mathrm{C} = \cap_{n\geq 1} A_n$ of I.

[3]) Giuseppe Peano was born on 27 Aug 1858 in Cuneo, Piemonte, Italy and died on 20 April 1932 in Turin, Italy. He invented space-filling curves in 1890, these are continuous surjective mappings from [0,1] onto the unit square. Hilbert, in 1891, described similar space-filling curves. It had been thought that such curves could not exist. Cantor had shown that there is a bijection between the interval [0,1] and the unit square but, shortly after, Netto had proved that such a bijection cannot be continuous. Peano's continuous space-filling curves cannot be 1-1 of course, otherwise Netto's theorem would be contradicted. Hausdorff wrote of Peano's result in "Grundzüge der Mengenlehre" in 1914: *This is one of the most remarkable facts of set theory.*

[4]) Fox, R. H., "On a Problem of S. Ulam Concerning Cartesian Products," *Fund. Math.* **34**, 278- 287 (1947) [MR 10, p.316].

† **Andrei Nikolaevich Tikhonov (1906-1993)** Andrei Nikolaevich Tikhonov was born on 30 Oct 1906 in Gzhatska, Smolensk, Russia and died in 1993. Like most Russian mathematicians there are different ways to transliterate Andrei Nikolaevich Tikhonov's name into the Roman alphabet. The most common way, other than Andrei Nikolaevich Tikhonov, is to write it as Andrey Nikolayevich Tychonoff.

Andrei Nikolaevich Tikhonov attended secondary school as a day pupil and entered the Moscow University in 1922, the year in which he completed his school education. His studied in the Mathematics Department of the Faculty of Mathematics and Physics at Moscow University and made remarkable progress, having his first paper published in 1925 while he was still in the middle of his undergraduate course.

This first work was related to results of ALEKSANDROV and URYSOHN on conditions for a topological space to be metrisable. However he did not stop there and continued his investigations in topology. By 1926 he had discovered the topological construction which is today named after him, the Tikhonov topology defined on the product of topological spaces. Aleksandrov, recalling that how he failed to appreciate the significance of Tikhonov's ideas at the time he proposed them, remembered:

... very well with what mistrust he met Tikhonov's proposed definition. How was it possible that a topology introduced by means of such enormous neighbourhoods, which are only distinguished from the whole space by a finite number of the coordinates, could catch any of the essntial characteristics of a topological product?

Tikhonov certainly had given the right definition and this idea, which was counterintuitive to even as great a topologist as Aleksandrov, allowed Tikhonov to go on and prove such important topological results as the product of any set of compact topological spaces is compact.

Few mathematicians have gained a worldwide reputation before they even start their research careers but this was essentially how it was for Tikhonov. His results on the Tikhonov topology of products were achieved before he graduated in 1927. With this impressive record he became a research student at Moscow University in 1927. It might be thought that someone who had clearly such an intuitive grasp of topological ideas would be only too pleased to use his talents in that area. Tikhonov, however, had equal talents for other areas of mathematics. The range of his work is summarised as:

We owe to Tikhonov deep and fundamental results in a wide range of topics in modern mathematics. His first-class achievements in topology and functional analysis, in the theory of ordinary and partial differential equations, in the mathematical problems of geophysics and electrodynamics, in computational mathematics and in mathematical physics are all widely known. Tikhonov's scientific work is characterised by magnificent achievements in very abstract fields of so- called pure mathematics, combined with deep investigations into the mathematical disciplines directly connected with practical requirements.

In fact Tikhonov's work led from topology to functional analysis with his famous fixed point theorem for continuous maps from convex compact subsets of locally convex topological spaces in 1935. These results are of importance in both topology and functional analysis and were applied by Tikhonov to solve problems in mathematical physics.

He defended his habilitation thesis in 1936 on Functional equations of Volterra type and their applications to mathematical physics. The thesis applied an extension of Emile Picard's method of approximating the solution of a differential equation and gave applications to heat conduction, in particular cooling which obeys the law given by Josef Stefan and Boltzmann . After successfully defending his thesis, Tikhonov was appointed as a professor at Moscow University in 1936 and then, three years later, he was elected as a Corresponding Member of the USSR Academy of Sciences. Tikhonov's approach to problems in mathematical physics is described as :

A characteristic of Tikhonov's research is to combine a concrete theme in natural science with investigations into a fundamental mathematical problem. In discussing some general problem in nature he always knows how to pick out a typical concrete physical problem and to give it a clear mathematical formulation. However, his mathematical investigations are never confined to the solution of a given concrete problem, but serve as the starting point for stating a general mathematical problem that is a broad generalisation of the first problem.

The extremely deep investigations of Tikhonov into a number of general problems in mathematical physics grew out of his interest in geophysics and electrodynamics. Thus, his research on the Earth's crust lead to investigations on well-posed Cauchy problems for parabolic equations and to the construction of a method for solving general functional equations of Volterra type. ...

Tikhonov's work on mathematical physics continued throughout the 1940s and he was awarded the State Prize for this work in 1953. However, in 1948 he began to study a new type of problem when he considered the behaviour of the solutions of systems of equations with a small parameter in the term with the highest derivative. After a series of fundamental papers introducing the topic, the work was carried on by his students.

Another area in which Tikhonov made fundamental contributions was that of computational mathematics:

Under his guidance many algorithms for the solution of various problems of electrodynamics, geophysics, plasma physics, gas dynamics, ... and other branches of the natural sciences were evolved and put into practice. ... One of the most outstanding achievemnets in computational mathematics is the theory of homogeneous difference schemes, which Tikhonov developed in collaboration with Samarskii....

In the 1960s Tikhonov began to produce an important series of papers on ill-posed problems. He defined a class of regularisable ill-posed problems and introduced the concept of a regularising operator which was used in the solution of these problems. Combining his computing skills with solving problems of this type, Tikhonov gave computer implementations of algorithms to compute the operators which he used in the solution of these problems. Tikhonov was awarded the Lenin Prize for his work on ill-posed problems in 1966. In the same year he was elected to full membership of the USSR Academy of Sciences.

Tikhonov's wide interests throughout mathematics led him to hold a number of different chairs at Moscow University, in particular a chair in the Mathematical Physics Faculty and a chair of Computational Mathematics in the Engineering Mathematics Faculty. He also became dean of the Faculty of Computing and Cybernetics at Moscow University. Tikhonov was appointed as Deputy Director of the Institute of Applied Mathematics of the USSR Academy of Sciences, a position he held for many years.

†† **Heinrich Franz Friedrich Tietze (1880-1964)** was born on 31 Aug 1880 in Schleinz (E of Neunkirchen), Austria and died on 17 Feb 1964 in Munich, Germany. Heinrich Tietze's father was Emil Tietze, the Director of the Geological Institute at the University of Vienna, and his mother was Rosa von Hauer, who was the daughter of the geologist Franz Ritter von Hauer. Tietze was a student at the Technische Hochschule in Vienna, starting his studies there in 1898. At Vienna he formed a close friendship with three other students of mathematics, PAUL EHRENFEST, HANS HAHN and GUSTAV HERGLOTZ. They were known as the 'inseparable four'.

It was his friend Herglotz who suggested that Tietze spend a year in Munich, and indeed he went there in 1902 to continue his studies. Returning to Vienna, Tietze was supervised during his doctoral studies by Gustav von Escherich and he was awarded his doctorate in 1904. Wirtinger, who had himself studied at Vienna, spent ten years at the University of Innsbruck before returning to a chair at the University of Vienna in 1905. He lectured on algebraic functions and their integrals in his first year back in Vienna, and Tietze attended these lectures and because of them formed an instant liking for topological notions which would from that time on be his main research topic.

Tietze submitted his habilitation thesis to Vienna in 1908 and this was on a topological topic considering topological invariants. From 1910 he was an extraordinary professor of mathematics at Brünn (today called Brno), and in 1913 he was promoted to ordinary professor. His career was interrupted, however, in 1914 by the outbreak of World War I.

At the start of the war Tietze was called up to serve in the Austrian army. He served for the duration of the war, returning to Brünn when hostilities had ended. The following year, in 1919, he accepted the chair of mathematics at the University of Erlangen. After six years in Erlangen, Tietze accepted a chair at the University of Munich. He remained in Munich for the rest of his life, retiring from his chair in 1950 but continuing his mathematical research almost up to the time of his death at age 83. Of course this means that Tietze spent the difficult years of Nazi control of Germany at Munich and this had many unfortunate consequences. CARATHEODORY was a colleague of Tietze's at Munich until he retired in 1938. The quest for a successor took from 1938 until 1944 and resulted in unbelievably complex political considerations. Tietze and his colleagues drew up a short list of three candidates to replace CARATHEODORY. These were HERGLOTZ, VAN DER WAERDEN and SIEGEL. However, all three were opposed by the Nazi professors at Munich for political reasons. LITTEN describes the arguments which involved considering the political reliability and the number of Jewish friends of the candidates.

Tietze contributed to the *foundations of general topology* and developed important work on *subdivisions of cell complexes*. The paper lists six books and 104 papers written by Tietze. The bulk of this work was carried out after he took up the chair at Munich in 1925.

Fundamental groups were introduced by POINCARE in 1895 and, in 1908, Tietze recognised that from the abelianised fundamental group of a space all the earlier invariants could be calculated. In that 1908 paper, Tietze produced a finite presentation for the fundamental group and invented the now well-known Tietze transformations to show that fundamental groups are topological invariants. The now famous Tietze transformations change one presentation of a finitely presented group to another presentation without changing the group which is defined by the presentation. It is possible to go from any given finite presentation of a group to any other using Tietze transformations.

In the same 1908 paper Tietze gives the first reference to the isomorphism problem for groups, namely: *if two groups are defined by finite presentations, is there an algorithm to decide whether they are isomorphic or not*? Of course Tietze gives the problem in the context of fundamental groups of topological spaces.

It is probably fair to say that VON DYCK initiated the study of combinatorial group theory but then Tietze made the first major step forward. Among the topics in topology which Tietze worked on were knot theory, Jordan curves and continuous mappings of areas. Tietze also wrote on map colouring and wrote a well known book Famous Problems of Mathematics. SEEBACH writes: *It shows his gift for representing even difficult mathematical questions in a very clear and impressive manner for interested people*.

Topics outside topology which Tietze worked on included ruler and compass constructions, continued fractions, partitions, the distribution of prime numbers, and differential geometry.

Tietze received many honours for his contributions. In particular he was elected a member of the Bavarian Academy of Sciences and served two terms (1934-42 and 1946-51) as Secretary to the Mathematics and Natural Sciences Division. He was also elected to the Austrian Academy of Sciences in 1959.