MA-231 Topology

10. Compact Spaces [1])

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10.1. (Compact Subsets) Let X be a topological space.

a). A subset E of X is compact if and only if every cover of E by open subsets of X has a finite subcover. — (**Remark:** Note that compactness is not a relative property; i.e. if E is compact, it is compact in whatever topological space it is embedded.)

b). The union of a finitely many compact subsets of X is compact. Arbitrary intersection of compact subsets of a Hausdorff topological space X is compact. — "Hausdorff" is necessary, even for intersection of two compact sets.

c). A compact subset of a non-Hausdorff topological space need not be closed.

d). The closure of a compact subset of a topological space need not be compact

e). In a Hausdorff topological space, disjoint compact subsets can be separated by disjoint sets. —(**Remark:** This is an illustration of the general rule: "*compact subsets behave like points.*")

10.2. (Nest characterisation of compactness) A topological space X is compact if and only if each nest [2]) of closed non-empty subsets in X has a non-empty intersection. (**Hint:** Suppose that each nest in X of closed non-empty subsets has a non-empty intersection and $\mathcal{F}$ is a family of closed subsets with the finite intersection property. Consider a maximal family $\mathcal{G}$ of closed subsets which contains $\mathcal{F}$ and has the finite intersection property and consider a maximal nest $\mathcal{M}$ in $\mathcal{G}$.)

10.3. (Maximal compact spaces[3])) A compact topological space X is called maximal compact if every strictly larger topology on X is noncompact.

a). A compact topological space X is maximal compact if and only if every compact subset is closed.

b). Every compact Hausdorff topological space is maximal compact and every maximal compact topological space is T_1. (**Remark:** Maximal compactness is like a separation axiom for compact spaces.)

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[1]) FRECHET was the first to use the term "compact". He used this term to those metric spaces in which every sequence of points contains a convegent subsequence or equivalently, in which every infinite set has a limit point. Applied to general topological spaces today, these define the *sequentially compact spaces* (see Exercise 10.4) and the *countably compact spaces* (see Exercise 10.3) respectively. HAUSDORFF first noticed that the present-day definition, in terms of the *Heine-Borel condition*, is equivalent in metric spaces to the definitions given above. It was left to ALXENDADROFF and URYSOHN to apply this definition to the general topological spaces; they called such spaces *bicompact*. Bicompactness won out over countable and sequential compactness when TYCHONOFF proved it was preserved in the passage to products. This result fails for both sequential compactness (see ???) and for countable compactness (???). That compactness could be described by using the finite intersection property was first noted by RIESZ.

[2]) A nest in a set X is a family of subsets which is linearly ordered by the inclusion.

[3]) Maximal compact spaces were studied by BALACHANDRAN (Professor RENUKA RAVINDRAN's teacher) and RAMANATHAN who were students of a well-known Indian topologist R. VAIDYANATHASWAMY who has written a book on set topology, which is one of the first books in english on the subject.

10.4. (Compact ordered spaces) Let $(X, \leq)$ be an ordered space, i.e. X is linearly ordered by the relation $\leq$ and the topology on X is generated by the subbase consisting of all sets of the form $\{x \in X \mid x < a\}$ and $\{x \in X \mid x > a\}$ for $a \in X$. Show that the following statements are equivalent :

(i) X is compact. (ii) X is *lattice complete*[4]) (iii) X is *Dedekind complete* [5]) and has a first and last element.

10.5. (Countably compact spaces) A topological space X is called countably compact if every countable open cover has a finite subcover.

a). A topological space X is countably compact if and only if each sequence has a cluster point. — (**Remark :** Hence, if and only if each sequence has a convergent subnet. This does not necessarily mean each sequence has a convergent subsequence. Spaces in which each sequence has a convergent subsequence are studied in the Exercise 10.4.)

b). A T_1-space is countably compact if and only if every infinite subset has a cluster point.

c). The product of a compact space and a countably space is countably compact. —(**Remark :** The result fails for two counably compact factors; Counter example!)

d). If $X_1, X_2, \ldots,$ is a countable family of first countable topological spaces, then $\prod_n X_n$ is countably compact if and only if each X_n is countably compact.

e). Continuous images and closed subspaces of countably compact spaces are countably compact.

f). For metric spaces, compactness and countable compactness are equivalent.

g). Let X be a countably compact topological space and let $x \in X$. If $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is a countable family of open sets in X such that $\overline{U_{n+1}} \subseteq U_n$ for all $U_n \in \mathcal{U}$ and $\cap_n U_n = \{x\}$, then $\mathcal{U}$ is a nhood base at x.

10.6. (Sequentially compact spaces) A topological space X is called sequentially compact if every sequencein X has a convergent subsequence.

a). Not every compact topological space is sequentially compact. (**Hint :** Consider an uncountable product of copies of $[0, 1]$.)

b). Every sequentially compact topological space is countably compact, but not every sequentially compact topological space is compact. —(**Remark :** hence, together with part a), sequentially compactness is neither stronger nor weaker than compactness; just diferent.)

c). A first countable topological space is sequentially compact if and only if it is countably compact. In particular, for metric spaces, sequential compactness is equivalent to compactness.

d). A second countable T_1 topological space is sequentially compact if and only if it is compact.

e). A countable product of sequentially compact topological spaces is sequentially compact. — (**Remarks :** It is also true, but dificult to prove, that the product of $\aleph_1$ sequentially compact topological spaces is countably compact. Assuming the continuum hypothesis, the product of any uncountable family of T_1-spaces, each having more than one point, is never sequentially compact.)

10.7. (Real-compact spaces[6])) A topological space X is called real-compact if it can be embedded as a closed subset of a product of copies of the real line $\mathbb{R}$.

a). Every compact Hausdorff space is real compact.

b). Every intersection of real-compact subsets of X is a real- compact.

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[4]) An ordered space $(X, \leq)$ is called lattice complete if each non-empty subset of X has a supremum and an infimum.

[5]) An ordered space $(X, \leq)$ is called Dedekind complete if every subset of X having upper bound has a least upper bound.

[6]) *Every compact Hausdorff topological space is Tychonoff and hence embeddable in some cube.* This makes it clear that a topological space is compact hausdorff if and only if it is embeddable as a closed subset of some product of copies of the closed unit interval $[0, 1]$ and leads to this generalisation of compactness.

c). Every product of real-compact spaces is real- compact.

10.8. (Pseudo-compact spaces) A topological space X is called pseudo-compact if every continuous real-valued function on X is bounded. Every countably compact topological space is pseudo-compact.

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† **Heinrich Eduard Heine (1821-1881)** Heinrich Eduard Heine was born on 16 March 1821 in Berlin, Germany and died on 21 Oct 1881 in Halle, Germany. Eduard Heine is important for his contributions to analysis. He went to lectures by Gauss and was taught by Dirichlet. Heine worked on Legendre polynomials, Lamé functions and Bessel functions. He is best remembered for the Heine-Borel theorem : *a subset of the reals is compact if and only if it is closed and bounded.* Heine also formulated the concept of uniform continuity. The paper written by P Dugac [*Sur la correspondance de Borel et le theoreme de Dirichlet- Heine- Weierstrass- Borel- Schoenflies- Lebesgue*, Arch. Internat. Hist. Sci. 39 (122) (1989), 69- 110.] analyses letters written by Heine and found in 1988 in the Institut Henri Poincaré. The second part of this paper covers the history of the Heine-Borel theorem and is summarised in the following review: The last half of the paper is devoted to a more systematic account of the gradual discovery and formulation of the so-called Heine- Borel theorem. It begins with the implicit use of the theorem in various proofs of the theorem stating that a continuous function on a closed, bounded interval is uniformly continuous. The first proof of this theorem was given by Dirichlet in his lectures of 1862 (published 1904) before Heine proved it in 1872. Dugac shows that Dirichlet used the idea of a covering and a finite subcovering more explicitly than Heine. This idea was also used by Weierstrass and Pincherle. Borel formulated his theorem for countable coverings in 1895 and Schönflies and Lebesgue generalized it to any type of covering in 1900 and 1898 (published 1904), respectively. Dugac shows that the story is in fact much more complicated and includes names such as Cousin, Thomae, Young, Vieillefond, Lindelöf. The priority questions are nicely illustrated with quotes from the correspondence between Lebesgue and Borel and other letters.

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†† **Pavel Samuilovich Urysohn (1898-1924)** Pavel Samuilovich Urysohn was born on 3 Feb 1898 in Odessa, Ukraine and died on 17 Aug 1924 in Batz-sur-Mer, France. Pavel Urysohn is also known as Pavel Uryson. His father was a financier in Odessa, the town in which Pavel Samuilovich was born. He came from a family descended from the sixteenth century Rabbi M Jaffe. It was a well- off family and Urysohn received his secondary education in Moscow at a private school there.

In 1915 Urysohn entered the University of Moscow to study physics and in fact he published his first paper in this year. Being interested in physics at this time it is not surprising that this first paper was on a physics topic, and indeed it was, being on Coolidge tube radiation. However his interest in physics soon took second place for after attending lectures by Luzin and Egorov at the University of Moscow he began to concentrate on mathematics. Urysohn graduated in 1919 and continued his studies there working towards his doctorate. Some authors write: *Luzin was a dynamic mathematician and it was he who persuaded Urysohn to stay on in order to study for a doctorate during* 1919-21. At this stage Urysohn was interested in analysis, in particular integral equations, and this was the topic of his habilitation. He was awarded his habilitation in June 1921 and, following this, became an assistant professor at the University of Moscow.

Urysohn soon turned to topology. He was asked two questions by Egorov and it was these which occupied him during the summer of 1921. The first question that Egorov posed was to find a general intrinsic topological definition of a curve which when restricted to the plane became Cantor's notion of a continuum which is nowhere dense in the plane. The second of Egorov's questions was a similar one but applied to surfaces, again asking for an intrinsic topological definition.

Now these were difficult questions which had been around for some time. It was not that Egorov had come up with new questions, rather he was giving the bright young mathematician Urysohn two really difficult problems in the hope that he might come up with new ideas. Egorov was not to be disappointed, for Urysohn attacked the questions with great determination. He did not sit still waiting for inspiration to strike, rather he tried one idea after another to see if it would give him the topological definition of dimension that he was looking for.

A holiday with other young Moscow mathematicians to the village of Burkov, on the banks of the river Kalyazmy near to the town of Bolshev, did not stop him trying to find the "right" definition of dimension. Quite the opposite, it was a good chance for him to think in congenial surroundings, and one morning near the end of August he woke up with an idea in his mind which he felt, even before working through the details, was right. Immediately he told his friend Aleksandrov about his inspiration.

Of course there was a lot of hard work after the moment of inspiration. During the following year Urysohn worked through the consequences building a whole new area of dimension theory in topology. It was an exciting time for the topologists in Moscow for Urysohn lectured on the topology of continua and often his latest results were presented in the course shortly after he had proved them. He published a series of short notes on this topic during 1922. The complete theory was presented in an article which Lebesgue accepted for publication in the Comptes rendus of the Academy of Sciences in Paris. This gave Urysohn an international platform for his ideas which immediately attracted the interest of mathematicians such as Hilbert.

Urysohn published a full version of his dimension theory in Fundamenta mathematicae. He wrote a major paper in two parts in 1923 but they did not appear in print until 1925 and 1926. Sadly Urysohn had died before even the first part was published. The paper begins with Urysohn stating his aim which was: *To indicate the most general sets that still merit being called "lines" and "surfaces" ...*

In fact Urysohn set out to do far more in this paper than to answer the two questions that Egorov had posed to him. As Crilly and Johnson write: Not only did he seek definitions of curve and surface, but also definitions of n-dimensional Cantorian manifold and hence of dimension itself. The dimension concept was, in fact, the centre of his attention.

Now, although Urysohn did not know of Brouwer's contribution when he worked out the details of his theory of topological dimension, Brouwer had in fact published on that topic in 1913. He had given a global definition, however, and this was in contrast to Urysohn's local definition of dimension. Another important aspect of Urysohn's ideas was the fact that he presented them in the context of compact metric spaces. After Urysohn's death, Aleksandrov argued that although Urysohn's definition of dimension was given for a metric space, it is, nevertheless, completely equivalent to the definition given by Menger for general topological spaces.

Urysohn visited Göttingen in 1923. His reports to the Mathematical Society of Göttingen interested Hilbert and while in Göttingen he learnt of Brouwer's contributions to the area made in the paper of 1913 to which we referred above. Urysohn spotted an error in Brouwer's paper regarding a definition of dimension while he was studying it in Göttingen and easily constructed a counter-example. He met Brouwer at the annual meeting of the German Mathematical Society in Marburg where both gave lectures and Urysohn mentioned Brouwer's error, and his counter-example, in his talk. It was an occasion which made Brouwer begin to think about topology again, for his interests had turned to intuitionism, the subject of his talk at Marburg.

In the summer of 1924 Urysohn set off again with Aleksandrov on a European trip through Germany, Holland and France. Again the two mathematicians visited Hilbert and, by 7 May, they must have left since Hilbert wrote to Urysohn on that day telling him his paper with Aleksandrov was accepted for publication in Mathematische Annalen (see below). This letter also thanks Urysohn for caviar he had given Hilbert, and expresses the hope that Urysohn will visit again the following summer.

They then met Hausdorff who was impressed with Urysohn's results. He also wrote a letter to Urysohn which was dated 11 August 1924. The letter discusses Urysohn's metrization theorem and his construction of a universal separable metric space. The construction of a universal metric space, containing an isometric image of any metric space, was one of Urysohn's last results. Like Hilbert, Hausdorff expressed the hope that Urysohn would visit again the following summer. Van Dalen writes about their final mathematical visit which was to Brouwer: *This time* [Urysohn and Aleksandrov] *visited Brouwer, who was most favourably impressed by the two Russians. He was particularly taken with Urysohn, for whom he developed something like the attachment to a lost son.* After this visit the two mathematicians continued their holiday to Brittany where they rented a cottage. Urysohn drowned in rough seas while on one of their regular swims off the coast.

Urysohn was not only an "inseparable friend" to Aleksandrov but the two collaborated on important publications such as Zur Theorie der topologischen Räume published in Mathematische Annalen in 1924. Urysohn's main contributions, in addition to the theory of dimension discussed above, are the introduction and investigation of a class of normal surfaces, metrization theorems, and an important existence theorem concerning mapping an arbitrary normed space into a Hilbert space with countable basis. He is remembered particularly for 'Urysohn's lemma' which proves the existence of a certain continuous function taking values 0 and 1 on particular closed subsets.

After Urysohn's death Brouwer and Aleksandrov made sure that the mathematics he left was properly dealt with. As van Dalen writes: *Brouwer was broken hearted. He decided to look after the scientific estate of Urysohn as a tribute to the genius of the deceased. Together with Aleksandrov he acquitted himself of this task.* Crilly and Johnson write : *Considering that he only had three years to devote to topology, he made his mark in his chosen field with brilliance and passion. He transformed the subject into a rich domain of modern mathematics. How much more might he there have been, had he not died so young?*