MA-231 Topology

11. Connected Spaces [1])

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Marie Ennemond Camille Jordan†
(1838-1922)

Arthur Moritz Schönflies††
(1853-1928)

11.1. (Examples on Connectedness) **a).** The sorgenfrey line is not connected.
b). Any infinite set with cofinite topology is connected. **c).** No countable subset of $\mathbb{R}$ is connected.

d). If $Y \subseteq \mathbb{R}^2$ is a countable set, then $\mathbb{R}^2 \setminus Y$ is path-connected.

e). Let $m \in \mathbb{N}^*$ and let X_m be the complement of m pairwise distinct affine lines in $\mathbb{R}^2$. How many connected components X_m (maximum and minimum?) can have? If the lines are replaced by circles of positive radii what is the answer? One can also analogously consider the problem in $\mathbb{R}^n$ by replacing affine lines (resp. circles) by affine hyperplanes (resp. (Euclidean) spheres).

11.2. Let $f : X \to Y$ be a continuous map between topological spaces. We say that f is locally constant, i.e. for every $x \in X$ there exists a nhood U of x such that the restriction $f|U$ is constant.

a). If X is connected and if f is locally constant, then f is constant. Moreover, if Y is a discrete topological space and if f is continuous, then f is constant.

b). If X is path-connected, then the image $f(X)$ of f is also path-connected.

c). If in addition f is open (or closed) with connected image $f(X)$ and connected fibres $f^{-1}(y)$, $y \in f(X)$, then X is also connected. (**Remark:** Does an analogous assertion also hold for path-connectedness?)

11.3. (Local Connectedness-- Local Pathwise Connected) Let X be a topological space. We say that X is locally pathwise connected (respectively locally connected) if each point of X has a nhood base consisting of pathwise connected respectively connected) subsets. — We should point out here that a subset A of X is pathwise connected if and only if any two points in A can be joined by a path *lying in* A.

a). The space $[0, 1) \cup (1, 2]$ is locally connected but not connected. The space X consisting of the vertical lines $x = 0$ and $x = 1$ in the plane, together with the horizontal line segments $\{(x, 1/n) \mid 1 \leq x \leq 1\}$ for $n = \pm1, \pm2, \ldots,$ and the unit interval $[0, 1]$ on the x-axis, is a typical example of connected space which are not locally connected, in fact, pathwise connected, but no point other than in $[0, 1]$ other than end points will have a base of connected nhoods. The Sorgenfrey line is not locally connected.

b). A connected, locally pathwise connected space X is pathwise connected. In particular, an open connected subset of $\mathbb{R}^n$ is pathwise connected.

c). A topological space X is locally connected if and only if each connected component of each open subset is open. In particular, the connected components of a locally connected space are open and closed. and a compact locally connected space has a finite number of connected components.

[1]) The modern notion of connectedness was proposed by JORDAN in 1893 and SCHOENFLIESZ and put on firm footing by RIESZ in 1906 with the use of subspace topologies. HAUSDORFF gave the first systematic account of the properties of connected sets. Connected components were introduced by HAUSDORFF. The notion of arcwise connecedness is much older than connectedness, having been used explicitly as early as the 1880's by WEIERSTRASS. Locally connected spaces were introduced by HAHN in 1914 and developed by TIETZE, KURATOWSKI and HAHN around 1920.

d). A topological space X is locally pathwise connected if and only if each path connected component of each open subset is open. A path component of X need not be closed. But if X is locally pathwise connected, the path components of X are open and closed.

e). If X is locally pathwise connected, then all connected components of X are open and equal to the path-connected components of X.

f). The continuous image of a locally connected space need not be locally connected. But every quotient of a locally connected space is locally connected. In particular, both continuous open images and continuous closed images of locally connected spaces are locally connected.

g). A non-empty product space $\prod_{i\in I} X_i$ is locally connected if and only if (1) each factor X_i is locally connected. (2) all but finitely many factors X_i are connected.

11.4. (Totally disconnected Spaces) Let X be a topological space. We say that X is *totally disconnected* if connected components of X are singletons. Equivalently, X is totally disconnected if and only if the only non-empty connected subsets of X are singeltons.

a). The Cantor set, the space $\mathbb{Q}$ of rationals, the space $\mathbb{R}\setminus\mathbb{Q}$ of irrationals and any discrete space are all totally disconnected.

b). The space $[X]$ of the connected components of X (with the quotient topology) is totally disconnected. (**Hint:** Show that every closed set $A\subseteq[X]$ with more than one point is not connected, for its inverse image in X has a decomposition into non-empty closed subsets which are unions of connected components of X.)

c). Every subspace of a totally disconnected space is totally disconnected. Every product of totally disconnected space is totally disconnected. Continuous image of a totally disconnected space need not be totally disconnected. (**Remark:** In fact, one of the amazing results in topology is: *Every compect metric space a continuous image of the cantor set.*)

11.5. Let V be a finite dimensional $\mathbb{R}$-vector space of dimension ≥ 2 and let $U_1,\ldots,U_r$ be affine subspaces in V of codimension ≥ 2. If G is a *domain* [2]) in V, then $G\setminus\bigcup_{\rho=1}^{r} U_\rho$ is also a domain in V. (**Hint:** It is enough to consider the case $r=1$. If $x\in G\setminus U_1$, then for every point $y\in G$ there exists a sequence of line segments from x to y, which lies completely in (eventually with the exception of the end point y) in $G\setminus U_1$.)

11.6. For a set X and a natural number n, let $\Delta_n=\Delta_n(X)$ denote the set of all those n-tuples $(x_1,\ldots,x_n)\in X^n$ whose components are not pairwise distinct. Then $X^n\setminus\Delta_n(X)$ is the set of all those n-tuples $(x_1,\ldots,x_n)\in X^n$, whose components are pairwise distinct.

a). If G is a domain in a finite dimensional $\mathbb{R}$-vector space V of dimension ≥ 2, then $G^n\setminus\Delta_n(G)$ is also a domain (in V^n) . (**Hint:** Use Exercise 11.6.)

b). How many connected components are there in $\mathbb{R}^n\setminus\Delta_n(\mathbb{R})$ respectively in $\mathbb{T}^n\setminus\Delta_n(\mathrm{S}^1)$, where $\mathbb{T}^n=(\mathrm{S}^1)^n$ is the n-dimensional torus?

11.7. The intersection of two open connected subsets in $\mathbb{R}^n$ is in general not connected, even if this intersection is non-empty. However: If $G_1, G_2\subseteq\mathbb{R}^n$ are open and connected with $G_1\cup G_2=\mathbb{R}^n$, then the intersection $G_1\cap G_2$ is a domain. (**Hint:** The proof for $n\geq 2$ is not so simple!)

11.8. (Connecedness in ordered spaces) Let $(X,\leq)$ be an ordered space. Then

a). X is connected if and only if it is Dedekind complete (see Exercise 10.4) and whenever $x<y$ in X, then $x<z<y$ for some z in X.

b). Every ordered space can be embedded in a connected ordered space. (**Hint:** First embed in a Dedekind complete ordered space. Then whenever $x<y$ in this space and no z exists with $x<z<y$, put a copy of $(0,1)$ between x and y.)

c). Let $\mathrm{I}:=[0,1]$ and $\{0,1\}$ have their usual orders and let $X:=\mathrm{I}\times\{0,1\}$ have the lexicographic order. Then X is Dedekind complete. What space results from applying the process in the part b) to X?

[2]) An open connected subset of a finite dimensional $\mathbb{R}$-vector space V is called a *domain* in V.

11.9. (Uses of Connecedness) Some Use of connectedness lies at the heart of most proofs that two spaces are not homeomorphic.

a). Use connectedness [3]) to show that X is not homeomoprhic to Y when:

(1) $X = \mathbb{R}$, $Y = \mathbb{R}^n$ for $n > 1$; (2) $X = [0, \infty)$, $Y = \mathbb{R}$; (3) $X = [0, 1]$, $Y = \mathbb{S}^1$;
(4) $X = \mathbb{S}^1$, $Y = \mathbb{S}^n$ for $n > 1$;

b). Any continuous map $f : [0, 1] \to [0, 1]$ has a fixed point, i.e. a point $x \in [0, 1]$ such that $f(x) = x$.

c). Let $P(x)$ be a polynomial (with real coefficients) of odd degree, then the equation $P(x) = 0$ has at least one real root.

d). Let f be a non-zero polynomial function on $\mathbb{K}^n$ with the zero-set $\mathrm{V}(f) := \{x \in \mathbb{K}^n \mid f(x) = 0\}$. In the case $\mathbb{K} = \mathbb{R}$, in general $\mathbb{R}^n \setminus \mathrm{V}(f)$ is not connected. For $\mathbb{K} = \mathbb{C}$: If $G \subseteq \mathbb{C}^n$ is a domain, then $G \setminus \mathrm{V}(f)$ is also a domain.

e). Let V be a finite dimensional norm linear $\mathbb{R}$-vector space with $\dim_{\mathbb{R}} V \geq 2$. If $f : V \setminus \{0\} \to \mathbb{R}$ is a continuous function, then there exists a $x \neq 0$ in V such that $f(x) = f(-x)$. (**Remark:** More generally: *If* $\dim_{\mathbb{R}} V \geq (n+1)$ *and* $f : V \setminus \{0\} \to \mathbb{R}^n$ *is continuous map, then there exists a* $x \neq 0$ *with* $f(x) = f(-x)$. — This is the famous theorem of BORSUK-ULAM.)

11.10. (Algebra based on connectedness[4]) — The group $H^0(X)$) Let X be a topological space. The set of all continuous maps from X to $\mathbb{Z}$ (ofcourse $\mathbb{Z}$ is discrete in the sense that each subset is open) is denoted by $H^0(X)$.[5])

a). For $f, g \in H^0(X)$, define $-f$ and $(f + g)$ by $(-f)(x) := -f(x)$ and $(f + g)(x) := f(x) + g(x)$ for $x \in X$. Then $-f$ and $f + g \in H^0(X)$ and $H^0(X)$ is an abelian group with respect to these operations.

b). A topological space X is connected if and only if every element of $H^0(X)$ is a constant map.

c). (Functorial properties of H^0) [6]) Let $f : X \to Y$ be a continuous map of topological spaces. Define $f^* : H^0(Y) \to H^0(X)$ by $f^*(g) := g \circ f$. Then show that

(i) f^* is a group homomorphism. (ii) If id_X is the identity map of X then $(\mathrm{id}_X)^*$ is the identity map of $H^0(X)$. (iii) If $g : Y \to Z$ is another continuous map then $(g \circ f)^* = f^* \circ g^*$.

d). Let $\alpha \in H^0(X)$ and $n \in \mathbb{N}^+$. If $n\alpha = 0$ then show that $\alpha = 0$, i.e. the group $H^0(X)$ is torsion free.

11.11. (Algebra based on path-conectedness — The group $\pi_0(X)$) Let $\sim$ be the relation on X defined by $x \sim y$ if there is a path in X joining x to y. Then $\sim$ is an equivalence relation on X. The quotient set $X/\sim$ is denoted by $\pi_0(X)$. The elements of $\pi_0(X)$ are called path-components of X. Evidently, X is path-connected if and only if $\pi_0(X)$ is singleton.

a). (Functorial properties of π_0)) [7]) Let $f : X \to Y$ be a continuous map of topological spaces. Then show that

(i) f induces a map $f_* : \pi_0(X) \to \pi_0(Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ \pi_0(X) & \xrightarrow{\;f_*\;} & \pi_0(Y) \end{array}$$

[3]) Note that in none of these cases can we distinguish between X and Y using any of the forms of compactness available to us.

[4]) Starting from a straightforward geometrical idea and then using algebraic methods to bulid apparatus on this is the characteristic method of algebraic topology. In this set we will develop algebra on the basis of the ideas of connectedness and path-connectedness.

[5]) The reason for the zero symbol in the notation is that H^0 is the first of sequence of analogous constructions, based on various dimensions. H^0 is the zero-dimensional case.

[6]) This exercise shows that H^0 is a contravariant functor from the category of topological spaces with continuous maps to the category of abelian groups with group homomorphisms.

[7]) This exercise shows that π_0 is a convariant functor from the category of topological spaces with continuous maps to the category of sets with the usual maps.

where $p_X : X \to \pi_0(X)$ and $p_Y : X \to \pi_0(Y)$ are the quotient maps.

(ii) If id_X is the identity map of X then $(\mathrm{id}_X)_*$ is the identity map of $\pi_0(X)$.

(iii) If $g : Y \to Z$ is another continuous map then $(g \circ f)_* = g_* \circ f_*$.

b). The continuous [8]) map $p_X : X \to \pi_0(X)$ induces (see Exercise 11.11-c)) a group isomorphism $p_X^* : H^0(X) \to H^0(\pi_0(X))$.

c). If X is compact and loally path-connected then show that $\pi_0(X)$ is a finite set.

d). Find a connected bounded closed subset of $\mathbb{R}^3$ which has infinite number of path-components.

11.12. Let X be a topological space and let U, V be disjoint open subsets of X. If $X = U \cup V$ then construct :

a). a bijective map $\pi_0(X) \to \pi_0(U) \cup \pi_0(V)$.

b). an isomorphism $H^0(X) \to H^0(U) \times H^0(V)$.

11.13. Let Y, Z be topological spaces and let $X = Y \times Z$ be the product space. Then

a). Is there in general a bijective map $\pi_0(X) \to \pi_0(Y) \times \pi_0(Z)$?

b). Is it true that $H^0(X)$ and $H^0(Y) \times H^0(Z)$ are in general isomorphic ?

11.14. Let $\mathrm{Maps}(\pi_0(X), \mathbb{Z})$ denote the set of all integer-valued functions on the set $\pi_0(X)$. The pointwise addition of functions gives $\mathrm{Maps}(\pi_0(X), \mathbb{Z})$ the structure of an abelian group. Then

a). There is a natural injective group homomorphism

$$c : H^0(X) \to \mathrm{Maps}(\pi_0(X), \mathbb{Z}) .$$

b). If X is locally path-connected then the natural group homomorphism c in the above part is an isomorphism.

c). Let $X := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$. Describe the image of the homomorphism c.

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† **Marie Ennemond Camille Jordan (1838-1922)** was born on 5 Jan 1838 in La Croix-Rousse, Lyon, France and died 22 Jan 1922 in Paris, France. Camille Jordan's father, Esprit- Alexandre Jordan (1800-1888), was an engineer who had been educated at the École Polytechnique. Camille's mother, Joséphine Puvis de Chavannes, was the sister of the famous painter Pierre Puvis de Chavannes who was the foremost French mural painter of the second half of the 19th century. Camille's father's family were also quite well known; a grand- uncle also called Ennemond-Camille Jordan (1771-1821) achieved a high political position while a cousin Alexis Jordan (1814-1897) was a famous botanist.

Jordan studied at the Lycée de Lyon and at the Collège d'Oullins. He entered the École Polytechnique to study mathematics in 1855. This establishment provided training to be an engineer and Jordan, like many other French mathematicians of his time, qualified as an engineer and took up that profession. Cauchy in particular had been one to take this route and, like Cauchy, Jordan was able to work as an engineer and still devote considerable time to mathematical research. Jordan's doctoral thesis was in two parts with the first part Sur le nombre des valeurs des foncyions being on algebra. The second part entitled Sur des periodes des fonctions inverses des intégrales des différentielles algebriques was on integrals of the form integral u dz where u is a function satisfying an algebraic equation f(u, z) = 0. Jordan was examined on 14 January 1861 by Duhamel, Serret and Puiseux. In fact the topic of the second part of Jordan's thesis had been proposed by Puiseux and it was this second part which the examiners preferred. After the examination he continued to work as an engineer, first at Privas, then at Chalon-sur-Saône, and finally in Paris. Jordan married Marie-Isabelle Munet, the daughter of the deputy mayor of Lyon, in 1862. They had eight children, two daughters and six sons.

From 1873 he was an examiner at the École Polytechnique where he became professor of analysis on 25 November 1876. He was also a professor at the Collège de France from 1883 although until 1885 he was at least theoretically still an engineer by profession. It is significant, however, that he found more time to undertake research when he was an engineer. Most of his original research dates from this period. Jordan was a mathematician who worked in a wide variety of different areas essentially contributing to every mathematical topic which was studied at that time.

Topology (called analysis situs at that time) played a major role in some of his first publications which were a combinatorial approach to symmetries. He introduced important topological concepts in 1866 built on his knowledge of Riemann's work in topology but not the work by Möbius for he was unaware of it. Jordan introduced the notion of homotopy of paths looking at the deformation of paths one into the other. He defined a homotopy group of a surface without explicitly using group terminology.

Jordan was particularly interested in the theory of finite groups. In fact this is not really an accurate statement, for it would be reasonable to argue that before Jordan began his research in this area there was no theory of finite groups. It was Jordan who was the first to develop a systematic approach to the topic. It was not until Liouville republished Galois's original work in 1846 that its significance was noticed at all. Serret, Bertrand and Hermite had attended Liouville's lectures on Galois theory and had begun to contribute to the topic but it was Jordan who was the first to formulate the direction the subject would take.

To Jordan a group was what we would call today a permutation group; the concept of an abstract group would only be studied later. To give an illustration of the way he tried to build up groups theory we will say a little about his contributions to finite soluble groups. The standard way the define such groups today would be to say that they are groups whose composition factors are abelian groups. Indeed Jordan introduced the concept of a composition series (a series of subgroups each normal in the preceding with the property that no further terms could be added to the series so that it retains that property). The composition factors of a group G are the groups obtained by computing the factor groups of adjacent groups in the composition series. Jordan proved the Jordan- Hölder theorem, namely that although groups can have different composition series, the set of composition factors is an invariant of the group.

Although the classification of finite abelian groups is straightforward, the classification of finite soluble groups is well beyond mathematicians today and for the foreseeable future. Jordan, however, clearly saw this as an aim of the subject, even if it was not one which might ever be solved. He made some remarkable contributions to how such a classification might proceed setting up a recursive method to determine all soluble groups of order n for a given n. A second major piece of work on finite groups was the study of the general linear group over the field with p elements, p prime. He applied his work on classical groups to determine structure of the Galois group of equations whose roots were chosen to be associated with certain geometrical configurations. His work on group theory done between 1860 and 1870 was written up into a major text Traité des substitutions et des équations algebraique which he published in 1870. This treatise gave a comprehensive study of Galois theory as well as providing the first ever group theory book. For this work he was awarded the Poncelet Prize of the Académie des Sciences. The treatise contains the 'Jordan normal form' theorem for matrices, not over the complex numbers but over a finite field. He appears not to have known of earlier results of this type by Weierstrass. His book brought permutation groups into a central role in mathematics and, until Burnside wrote his famous group theory text nearly 30 years later, this work provided the foundation on which the whole subject was built. It would also be fair to say that group theory was one of the major areas of mathematical research for 100 years following Jordan's fundamental publication.

Jordan's use of the group concept in geometry in 1869 was motivated by studies of crystal structure. He considered the classification of groups of Euclidean motions. His work had gained him a wide international reputation and both Sophus Lie and Felix Klein visited him in Paris in 1870 to study with him. Jordan's interest in groups of Euclidean transformations in three dimensional space influenced Lie and Klein in their own theories of continuous and discontinuous groups. The publication of Traité des substitutions et des équations algebraique did not mark the end of Jordan's contribution to group theory. He went on over the next decade to produce further results of fundamental importance. He studied primitive permutation groups and proved a finiteness theorem. He defined the class of a subgroup of the symmetric group to be c > 1 if c was the smallest number such that the subgroup had an element moving c points. His finiteness theorem showed that for a given c there are only finitely many primitive groups with class c other than the symmetric and alternating groups.

Generalising a result of Fuchs on linear differential equations, Jordan was led to study the finite subgroups of the general linear group of n cross n matrices over the complex numbers. Although there are infinite families of such finite subgroups, Jordan found that they were of a very specific group theoretic structure which he was able to describe. Another generalisation, this time of work by Hermite on quadratic forms with integral coefficients, led Jordan to consider the special linear group of n cross n matrices of determinant 1 over the complex numbers acting of the vector space of complex polynomials in n indeterminates of degree m.

Jordan is best remembered today among analysts and topologists for his proof that a simply closed curve divides a plane into exactly two regions, now called the Jordan curve theorem. It was only his increased understanding of mathematical rigour which made him realise that a proof of such a result was necessary. He also originated the concept of functions of bounded

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[8]) $\pi_0(X)$ has the quotient topology with respect to p_X

variation and is known especially for his definition of the length of a curve. These concepts appears in his Cours d'analyse de l'École Polytechnique first published in three volumes between 1882 and 1887. The second edition appeared in 1893 while the Jordan curve theorem appeared in the third edition of the text which appeared between 1909 and 1915.

Of course by 1882, when the first volume was published, Jordan was lecturing at the École Polytechnique and the book was written as a text for the students there. In some respects this is a little strange since it is a rigorous analysis text built on top of the attempts to put the topic on a firm foundation begun by Cauchy and given considerable impetus by Weierstrass. However, the courses at the École Polytechnique were supposed to train students to become civil and military engineers and this does not seem to be the approach which one would take trying to teach applications of the calculus to engineers. There had been a tradition of rigorous analysis at the École Polytechnique begun, of course, by Cauchy himself. Jordan was aware that his work was at a level that would be somewhat inappropriate for engineering students for he once said to Lebesgue that he called it "École Polytechnique analysis course" since: ... *one puts that on the cover to please the publisher...*

Gispert-Chambaz contrasts the way that topological concepts are treated by Jordan in the first and second editions of the book. In the first addition most of the topological concepts are dealt with in an supplement to Volume 3. However between the editions Jordan had taught more advanced courses on analysis at the Collège de France and this may have influenced him to put set topology right up front in the second edition. In this respect one can see the second edition as setting a tone for analysis textbooks which continues today.

Among Jordan's many contributions to analysis we should also mention his generalisation of the criteria for the convergence of a Fourier series. The Journal de Mathématiques Pure et Appliquées was a leading mathematical journal and played a very significant part in the development of mathematics throughout the 19th century. It was usually known as the Journal de Liouville since Liouville had founded the journal in 1836. Liouville died in 1882 and in 1885 Jordan became editor of the Journal, a role he kept for over 35 years until his death. In 1912 Jordan retired from his positions. The final years of his life were saddened, however, because of World War I which began in 1914. Between 1914 and 1916 three of his six sons were killed in the war. Of his three remaining sons, Camille was a government minister, Edouard was a professor of history at the Sorbonne, and the third son was an engineer. Among the honours given to Jordan was his election to the Académie des Sciences on 4 April 1881. On 12 July 1890 he became an officer of the Légion d'honneur. He was the Honorary President of the International Congress of Mathematicians at Strasbourg in September 1920.

Finally we should note some rather confusing facts. Although given Jordan's work on matrices and the fact that the Jordan normal form is named after him, the Gauss-Jordan pivoting elimination method for solving the matrix equation Ax= b is not. The Jordan of Gauss-Jordan is Wilhelm Jordan (1842 to 1899) who applied the method to finding squared errors to work on surveying. Jordan algebras are called after the German physicist and mathematician Pascual Jordan (1902 to 1980).

†† **Arthur Moritz Schönflies (1853-1928)** was born on 17 April 1853 in Landsberg an der Warthe, Germany (now Gorzów, Poland) and died on 27 May 1928 in Frankfurt am Main, Germany. Arthur Schönflies was a student at the University of Berlin from 1870 until 1875 working under Kummer and Weierstrass. He obtained a doctorate from Berlin in 1877 and the following year he obtained a post as a teacher at a school in Berlin. In 1880 he went to Colmar in Alsace to teach. He then wrote his Habilitation thesis which he presented to Göttingen, the qualification being awarded in 1884. Klein worked to set up a chair of applied mathematics at Göttingen and in 1892 Schönflies was appointed to this chair.

He left Göttingen in 1899 to take up a chair at Königsberg, then in 1911 he became professor at the Academy for Social and Commercial Sciences in Frankfurt. This Academy became a University in 1914. Schönflies ended his career at the University of Frankfurt where he served as professor from 1914 until 1922 being rector of the University in the session 1920- 21. Schönflies worked first on geometry and kinematics but became best known for his work on set theory and crystallography. Klein suggested the problem of finding the crystallographic space groups in the late 1880s. By 1891 he had found the complete list of 230 such groups. His presentation of crystallographic space groups published in 1892 used the latest aspects of group theory and became a classic on the subject.

In fact the classification of the crystallographic space groups was made independently by E S Fedorov. Schönflies corresponded with Fedorov and corrected some minor errors in his classification. He republished his classification in 1923 and in the same year he published a book on crystallography. In around 1895 Schönflies turned his attention towards set theory and topology. He wrote many works which were important at the time they were published but they were rather superseded by Hausdorff's Grundzüge der Mengenlehre in 1914. Three important papers on plane topology proved the topological invariance of the dimension of the square. His work contains gaps and errors which were investigated by Brouwer who made some deep discoveries from studying these errors. Schönflies also wrote on kinematics and projective geometry. He wrote textbooks on descriptive geometry and analytic geometry and a calculus textbook jointly with Nernst. In 1895 Schönflies edited Plücker's complete works.