The Bicharacteristic Theorem

A theorem very useful for a deep understanding of solutions of hyperbolic systems and in developing numerical methods for their solutions

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The One-dimensional Wave Equation

$$u_{tt} - a^2\ u_{xx} = 0,\ \ a = \text{constant}\ \ > 0. \tag{1}$$

The characteristic PDE of (1) is

$${\varphi_t}^2 - a^2{\varphi_x}^2 = 0. \tag{2}$$

The equation (2) contains two first order equations

$$\varphi_t + a\varphi_x = 0 \quad \textit{and}\ \ \varphi_t - a\varphi_x = 0. \tag{3}$$

Characteristic equations of these two characteric PDEs give the two families of characteristic curves

$$x - at = constant = \xi\ \ \ say\ ; \quad and\ \ x + at = constant = \eta\ \ \ say\ . \tag{4}$$

I am sure you will not be surprised to see too many use of the word characteristic, because you have been taught this. But the wave equation in 3 space dimensions will be an eye opener.

Solution of One-dimensional Wave Equation.

In terms of characteristic variables ξ and η in (4) the equation (1) becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0, \tag{5}$$

which immediately gives the general solution of (1) in the form

$$u = f(x - at) + g(x + at). \tag{6}$$

Given an initial value problem or Cauchy Problem for (1), we can use (5) to give an explicit solution.

This is not our aim. We wish to say that the solution of the wave equation in multi-space dimensions is very complex.

One-dimensional wave equation contd..

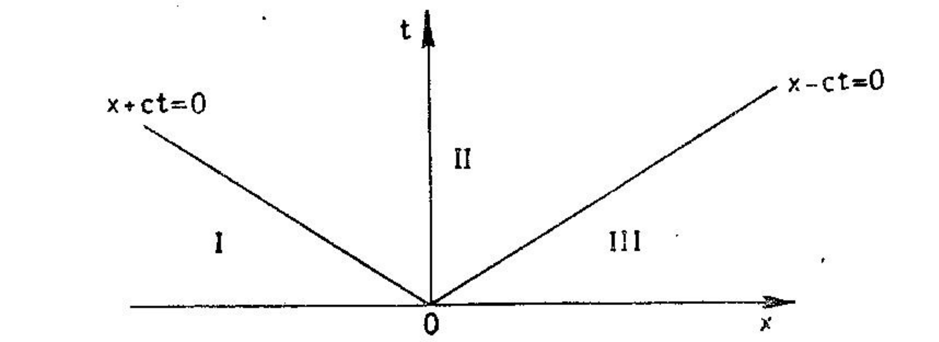


Figure: 1 Characteristic curves through $(0,0)$. Replace c by a here.

These characteristics are solutions of the FO Characteristic PDE of (2) and also of the Characteristic ODEs $\frac{dx}{dt} = \pm a$ of the (2).
Note that characteristics of characteristic PDE are also called characteristics.

Wave Equation in m-space Dimensions

Consider the wave equation in $\mathbb{R}^{m+1}$:

$$u_{tt} - a^2 \, \Delta u = 0, \tag{7}$$

where $\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$, $a = \text{constant} > 0$.

- We need to distinguish between two gradient operators:

$$\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}) \quad and \quad \nabla_{(\boldsymbol{x},t)} = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial t}), \tag{8}$$

- Characteristic PDE of (7) is a first order PDE for the characteristic surfaces $\Omega : \varphi(\mathbf{x}, t) = \text{constant}$

$$Q(\nabla\varphi, \varphi_t) := \varphi_t^2 - a^2 \, |\nabla.\varphi|^2 = 0 \tag{9}$$

Note symbol $:=$ means defined by.

Wave Equation in m-space Dimensions ... continued

- Equation (8) is equivalent to two first order PDEs:

$$\tilde{Q}_1(\nabla\varphi, \varphi_t) := \varphi_t + a\,|\nabla\varphi| = 0 \tag{10}$$

and

$$\tilde{Q}_2(\nabla\varphi, \varphi_t) := \varphi_t - a\,|\nabla\varphi| = 0, \tag{10a}$$

where $|\nabla\varphi| = \surd(\varphi_{x_1}^2 + \varphi_{x_2}^2 + ... + \varphi_{x_m}^2)$.

- A particular solution of (10) representing the forward characteristic conoid with vertex at the origin is

$$\varphi := t - a\sqrt{x_1^2 + x_2^2 + \cdots + x_{m-1}^2 + x_m^2} = 0. \tag{11}$$

Wave Equation in m-space Dimensions ... continued

- The characteristic ODEs of (10),

  i.e. $\tilde{Q}_1(\nabla\varphi, \varphi_t) := \quad \varphi_t + a\,|\nabla\varphi| = 0$ are

$$\frac{dt}{d\sigma} = \tilde{Q}_{1\varphi_t} = 1, \qquad \frac{dx_\alpha}{d\sigma} = \tilde{Q}_{1\varphi_{x_\alpha}} = a\frac{\varphi_{x_\alpha}}{|\nabla\varphi|} \tag{12}$$

$$\frac{d\varphi_t}{d\sigma} = -\tilde{Q}_{1t} = 0, \qquad \frac{d\varphi_{x_\alpha}}{d\sigma} = -\tilde{Q}_{1x_\alpha} = 0. \tag{13}$$

- In terms of unit normal $\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$ of the wavefront, the equations (12) and (13) become

$$\frac{d\boldsymbol{x}}{dt} = a\boldsymbol{n}, \qquad \frac{d\boldsymbol{n}}{dt} = 0, \tag{14}$$

  which give totality of all straight lines (not parallel to $t = 0$) in space-time.

Wave Equation in m-space Dimensions $\cdots$ Conti.

Thus the characteristics of the characteristic PDE (**i.e, a new name bicharacteristics)** passing through the origin are

$$\boldsymbol{x} = a\boldsymbol{n}t. \tag{15}$$

When $\boldsymbol{n}$ varies the lines given by (15) envelop the characteristic conoid (14) with vertex at the origin.

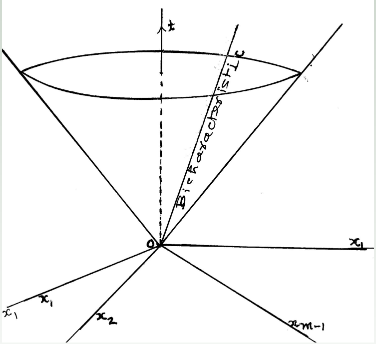


Figure: 2 Characteristic conoid through $(\mathbf{0}, 0)$ of wave equation m-space dimensions.

Characteristic Coinoid of the Wave Equation

- Bicharacteristics are always one-dimensional, i.e. curves in space-time.

- For $m = 2$, the characteristic coinoid is a 2-D manifold and bicharacteristic on it form a one-parameter family of curves, since we can choose $\boldsymbol{n} = (\cos\theta, \sin\theta)$.

- For $m = 3$, the characteristic coinoid is a 3-D manifold in 4-D space-time and bicharacteristic on it form a two-parameter family of curves. You can easily
  (i) visualise the geometry of a section of the characteristic coinoid by $t = constant$ and
  (ii) find out a pair of parameters of the bicharacteristics.

- For a general m, the characteristic coinoid is a m-D manifold and bicharacteristic on it form a $(m-1)$-parameter family of curves.

Bicharacteristics

- For 1-D wave equation characteristic curves and the characteristic curves of the characteristic PDE are the same.
- This is not so for the wave equation in multi-space dimensions and it will be wise mathematically to adopt a new name **bicharacteristic**.
- In 1970 I could find reference to this word only in Courant and Hilbert [2] as "Lemma on Bicharacteristic Directions".
- It attracted my attention and I started using this word in my publications since 1973.
- It is surprising that, the comprehensive book by C. M. Dafermos [1] running into the third edition in 2016 and which refers to our work in 3 places, does not have this word.

References to Bicharacteristics

You can see details of bicharacteristics, significance and its use in applications to physics in our 4 books, (apart from C&H).

PP & RR 1984 [3]

PP 1993 research monograph [4]

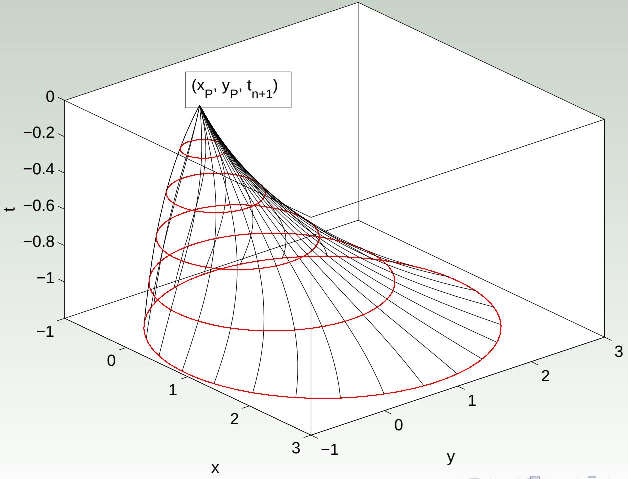
PP 2001 research monograph [5]

PP 2018 research monograph [6]

See also my article on the webpage:
http://www.math.iisc.ernet.in/~prasad/prasad/MomentsOfSupremeHappinessAndSatifactionResearch.pdf

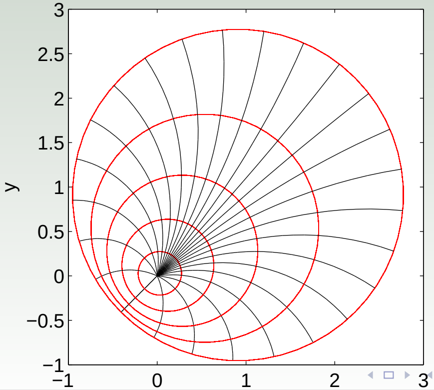
Another Example of Bicharacteristics - Fig 3: Sections of the characteristic conoid are shown in red and bicharacteristics in black

Two-D wave equation $u_{tt} - a(u_{xx} + u_{yy}) = 0$
with $a(x, y) = a_0 + a_1(x - x_0) + a_2(y - y_0)$



Backward Moving Wavefronts and Rays by Projection of Previous Figure on $\boldsymbol{x}$-space

From Arun, K.R., Kraft, M., Lukacov'a Medvidov'a, M., and Phoolan Prasad, *Finite volume evolution Galerkin method for hyperbolic conservation laws with spatially varying flux functions*, Journal of Computational Physics, **228**, 565-590, 2009. Wavefronts are shown in red and rays in black.



System of 1st Order PDE in m-space Dimensions

- Consider a system of n first order PDE in $(\boldsymbol{x}, t) \in \mathbb{R}^{m+1}$- (the $m+1$ dimensional space-time):

$$A(\boldsymbol{x}, t, \boldsymbol{u})\boldsymbol{u}_t + B^{(\alpha)}(\boldsymbol{x}, t, \boldsymbol{u})\boldsymbol{u}_{x_\alpha} + \boldsymbol{C}(\boldsymbol{x}, t, \boldsymbol{u}) = 0, \tag{16}$$

where the sum is over α is on $(1, 2, \cdots, m)$; $\boldsymbol{u} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B^{(\alpha)} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C} \in \mathbb{R}^n$.

- Characteristic PDE of (16) is

$$Q(\boldsymbol{x}, t; \boldsymbol{\nabla}\varphi, \varphi_t) \equiv \det(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}) = 0, \tag{17}$$

where we have not shown the dependence of Q on a known solution $\boldsymbol{u}(\boldsymbol{x}, t)$.

Characteristic Manifold Ω and Wavefront Ω_t

- A surface satisfying (17) is a characteristic manifold in space-time:

  $$\Omega := \varphi(\boldsymbol{x}, t) = 0 \qquad (17a.)$$

  Its section by the plane $t = constant$ is also represented by $\varphi(\boldsymbol{x}, t) = 0$, where $t =$ is kept constant.

- Projection of the section by $t = constant$ on the physical space $\boldsymbol{x}$ is a wavefront, see slides 12 & 13. It is denoted by

  $$\Omega_t := \varphi(\boldsymbol{x}, t) = 0 \qquad t = \quad constant. \qquad (17b)$$

  Unit normal $\boldsymbol{n}$ of the wavefront is given by

  $$\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}. \qquad (17c)$$

- Velocity of propagation of the wavefront is given by

  $$c = -\varphi_t/(|\nabla\varphi|). \qquad (17d)$$

Hyperbolic System of 1st Order PDE in m-space Dimensions

- In terms of the he unit normal $\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$ of the wavefront Ω_t and its the normal velocity $C = -\frac{\varphi_t}{|\nabla\varphi|}$, the PDE (17) becomes

$$\det\left[n_\alpha B^{(\alpha)} - CA\right] = 0 \tag{18}$$

- **Definition:** We define the system (16) as hyperbolic in a domain $D \in \mathbb{R}^{m+1}$ with t as time-like variable if, given an arbitrary unit vector $\boldsymbol{n}$, the nth degree characteristic equation (18) in C has n real roots (called eigenvalues) $c_1, c_2, \ldots, c_n$ and eigenspace is complete at each point of D.

- We assume that at each point of D and for all $\boldsymbol{n}$

$$c_1 \leq c_2 \leq c_3 \leq \ldots \leq c_n. \tag{19}$$

This means that the system (16) is hyperbolic with characteristics of uniform multiplicity.

Left and Right Eigen Vectors of a Hyperbolic System

- We denote left and right eigenvectors satisfying

$$\boldsymbol{\ell}^{(i)}\ (n_\alpha B^{(\alpha)}) = c_i \boldsymbol{\ell}^{(i)} A, \quad (n_\alpha B^{(\alpha)}) \boldsymbol{r}^{(i)} = c_i A \boldsymbol{r}^{(i)}. \tag{20}$$

  by $\boldsymbol{\ell}^{(i)}$ and $\boldsymbol{r}^{(i)}$.

- Suppose an eigenvalue $c_i(\boldsymbol{x}, t, \boldsymbol{u}, \boldsymbol{n})$ is repeated p_i times in (19), completeness of eigenspace at each point of D implies that the number of linearly independent left eigenvectors (and hence also right eigenvectors) corresponding to c_i is p_i.

- Each of the left and right eigenvectors $\boldsymbol{\ell}^{(i)}$, $\boldsymbol{r}^{(i)}$ is unique except for a scalar multiplier.

- We normalise the eigenvectors such that

$$|\boldsymbol{\ell}^{(i)}| = 1 \ \ \textit{and} \ \ |\boldsymbol{r}^{(i)}| = 1. \tag{20a}$$

  Now these eigenvectors are unique.

Characteristic PDE for a Particular Eigen Value c_i

For simplicity in notation, we drop the subscript i from c_i and superscript (i) from $\boldsymbol{\ell}^{(i)}$ and $\boldsymbol{r}^{(i)}$.

- We take the first relation in (20) (dropping the subscript i and superscript (i)), post-multiply by $\boldsymbol{r}$ and use $n_\alpha = \varphi_{x_\alpha}/|\boldsymbol{\nabla}\varphi|$ and $c = -\varphi_t/|\boldsymbol{\nabla}\varphi|$.
- This gives the relation satisfied by ϕ_t and ϕ_{x_α} in the form (another form of the characteristic PDE but only for one mode c)

$$\tilde{Q}(\boldsymbol{x}, t, \boldsymbol{\nabla}\varphi, \varphi_t) := (\boldsymbol{\ell} A \boldsymbol{r})\varphi_t + (\boldsymbol{\ell} B^{(\alpha)} \boldsymbol{r})\varphi_{x_\alpha} = 0. \tag{21}$$

- This is similar to one of the the characteristic PDEs in (10) for the forward and (10a) for backward characteristic conoids for the wave equation.

Characteristic ODE of Characteristic PDE (21)

The characteristic ODE are the Charpit's ODEs of (21) are

$$\frac{dt}{d\sigma} = \frac{1}{2}\tilde{Q}_{\varphi_t}, \qquad \frac{dx_\alpha}{d\sigma} = \frac{1}{2}\tilde{Q}_{\varphi_\alpha} \tag{22}$$

and

$$\frac{d\varphi_t}{d\sigma} = -\frac{1}{2}\tilde{Q}_t, \quad \frac{d\varphi_{x_\alpha}}{d\sigma} = -\frac{1}{2}\tilde{Q}_{x_\alpha} \tag{23}$$

where we need to impose a condition $\tilde{Q}(\boldsymbol{x}, t, \boldsymbol{\nabla}\varphi, \varphi_t) = 0$.

The equations (22)and (23) give the bicharacteristic curves on the characteristic manifold Ω, given by $\varphi(\boldsymbol{x}, t) = 0$.

From theory of first order PDE, it follows that the surface Ω is generated by a family of bicharacteristic curves. Thus the bicharacteristic direction given by (22) is tangential to any Ω.

Characteristic ODE of Characteristic PDE (21) conti···

- Since $\boldsymbol{\ell}$ and $\boldsymbol{r}$ depend on φ_t and φ_{x_α} in very involved way, the partial derivatives of $\tilde{Q}$ on the right hand sides of (22) and (23) will be quite complicated.

However the following lemma makes the results simple.

Lemma 1: The partial derivatives of $\tilde{Q}$ simplify as

$$\tilde{Q}_{\varphi_t} = (\boldsymbol{\ell} A \boldsymbol{r}), \qquad \tilde{Q}_{\varphi_{x_\alpha}} = (\boldsymbol{\ell} B^{(\alpha)} \boldsymbol{r}). \tag{24}$$

$$\tilde{Q}_t = (\boldsymbol{\ell} A_t \boldsymbol{r})\varphi_t + (\boldsymbol{\ell} B_t^{(\alpha)} \boldsymbol{r})\varphi_{x_\alpha}, \quad \tilde{Q}_{x_\alpha} = (\boldsymbol{\ell} A_{x_\alpha} \boldsymbol{r})\varphi_t + (\boldsymbol{\ell} B_{x_\alpha}^{(\beta)} \boldsymbol{r})\varphi_{x_\beta} \tag{25}$$

Proof of Lemma on Bicharacteristic ODEs

Proof of the lemma 1: We proof of the first part of the (24). Other parts of the Lemma 1 fallow in similar way.

$$\tilde{Q}_{\varphi_t} = \boldsymbol{\ell}_{\varphi_t}\{A\boldsymbol{r}\varphi_t + (B^{(\alpha)}\boldsymbol{r})\varphi_{x_\alpha}\} + \boldsymbol{\ell}A\boldsymbol{r}$$

$$+ \{\boldsymbol{\ell}A\varphi_t + (\boldsymbol{\ell}B^{(\alpha)}\varphi_{x_\alpha}\}\boldsymbol{r}_{\varphi_t} \qquad (26)$$

Using (17c,d), i.e $\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$ and $c = -\varphi_t/(|\nabla\varphi|$

$$\boldsymbol{\ell}_{\varphi_t}\{A\boldsymbol{r}\varphi_t + (B^{(\alpha)}\boldsymbol{r})\varphi_{x_\alpha}\} = |\nabla\varphi|\boldsymbol{\ell}_{\varphi_t}\{-cA + B^{(\alpha)}n_\alpha\}\boldsymbol{r},$$

which vanishes due to the second equation in (20). Similarly the third term in (26) disappears and we get the first equation in (24).

Bicharacteristic Equations - conti. $\cdots$

From equations (22 - 25) we get

$$\frac{dx_\alpha}{dt} = \frac{(\boldsymbol{\ell} B^{(\alpha)} \boldsymbol{r})}{(\boldsymbol{\ell} A \boldsymbol{r})} = \chi_\alpha \quad , say \tag{27}$$

$$\frac{d\varphi_{x_\alpha}}{dt} = -\frac{(\boldsymbol{\ell} A_{x_\alpha} \boldsymbol{r})\varphi_t + (\boldsymbol{\ell} B^{(\beta)}_{x_\alpha} \boldsymbol{r})\varphi_{x_\beta}}{(\boldsymbol{\ell} A \boldsymbol{r})} = |\nabla\varphi| \frac{c(\boldsymbol{\ell} A_{x_\alpha} \boldsymbol{r}) - (\boldsymbol{\ell} B^{(\beta)}_{x_\alpha} \boldsymbol{r}) n_\beta}{(\boldsymbol{\ell} A \boldsymbol{r})} \tag{28}$$

We shall transform (28) for a physically realistic variable, namely normal $\boldsymbol{n}$ to the wavefront Ω_t.

$$\frac{dn_\alpha}{dt} = \frac{d}{dt}\left\{\frac{\varphi_{x_\alpha}}{|\nabla\varphi|}\right\} = \frac{1}{|\nabla\varphi|}\left\{\frac{d\varphi_{x_\alpha}}{dt} - n_\alpha n_\beta \frac{d\varphi_{x_\beta}}{dt}\right\}$$

$$= \frac{n_\beta}{|\nabla\varphi|}\left\{n_\beta \frac{d\varphi_{x_\alpha}}{dt} - n_\alpha \frac{d\varphi_{x_\beta}}{dt}\right\}, \quad using \;\; n_\beta n_\beta = 1. \tag{29}$$

Bicharacteristic Equations - conti. $\cdots$

- Substituting expressions from (28) in (29) and after rearranging terms we get

$$\frac{dn_\alpha}{dt} = -\frac{1}{\boldsymbol{\ell} A \boldsymbol{r}} \boldsymbol{\ell} \left\{ n_\beta \left(n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta^\alpha_\beta} - c \frac{\partial A}{\partial \eta^\alpha_\beta} \right) \right\} \boldsymbol{r} = \psi_\alpha, \quad say, \tag{30}$$

where

$$\frac{\partial}{\partial \eta^\alpha_\beta} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta}. \tag{31}$$

- This operator is tangential to the wavefront Ω_t which is projection of a section of the characteristic manifold Ω and hence also to Ω.
- The vector $\boldsymbol{\chi} = (\chi_1, \cdots, \chi_m)$ is called the ray velocity and the operator tangential to Ω

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_\alpha \frac{\partial}{\partial x_\alpha} \tag{32}$$

represents differentiation in a bicharacteristic direction.

Dynamics of Wave Equations

- Dynamics is an important aspect of wave equations and "almost" independent of their kinematics, which we have been discussing so far.
- For the wave equation in multi-space dimensions, a good discussion is available in our book PP-RR (1984) [3], first edition available freely on *http://www.math.iisc.ernet.in/~prasad/prasad/book/PP-RR_PDE_book_1984.pdf*
- It is good to study the last chapter, specially Part-B of the Chapter 3. It has a very good introduction to bicharacteristics.
- The other three research monographs [4, 5, 6] also contain these aspects. A part of [6] is available on *http://www.math.iisc.ernet.in/~prasad/prasad/book/2018_PP_book_cover_&_first_12_pages.pdf*

Dynamics of 1-D Wave Equation

- We note

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u = \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)u. \qquad (33)$$

- Define characteristic varaibles $r = u_t + au_x$ and $s = u_t - au_x$, then

$$\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)r = 0 \text{ and } \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)s = 0. \qquad (34)$$

- These two equations are compatibility conditions and contain the dynamics of the solution.
- The first one tells that the part r moves unchanged along the characteristic with velocity a. Similarly part s moves unchanged along the characteristic with velocity $-a$.

Dynamics of a Hyperbolic System of $1st$ Order PDEs

- Pre-multiplying (16) by $\boldsymbol{\ell}$, we get

$$\boldsymbol{\ell} A \boldsymbol{u}_t + \boldsymbol{\ell} B^{(\alpha)} \boldsymbol{u}_{x_\alpha} + \boldsymbol{\ell} \boldsymbol{C} = 0. \tag{35}$$

  From general theory it follows that in this scalar equation every dependent variable is differentiated in a tangential direction on Ω.

- Therefore (35) represents a compatibility condition on the characteristic surface Ω.

- Using the expression (32) for $\frac{d}{dt}$, the time rate of change along a bicharacteristic, in (35) we rewrite it in the form

$$\boldsymbol{\ell} A \frac{d\boldsymbol{u}}{dt} + \boldsymbol{\ell}(B^{(\alpha)} - \chi_\alpha A) \frac{\partial \boldsymbol{u}}{\partial x_\alpha} + \boldsymbol{\ell} \boldsymbol{C} = 0. \tag{36}$$

Dynamics of a Hyperbolic System of $1st$ Order PDEs Conti. $\cdots$

- We write (36) in the form

$$l_i A_{ij} \frac{du_j}{dt} + \tilde{\partial}_j u_j + l_i C_i = 0 \tag{37}$$

where $\tilde{\partial}_j$ on u_j, a special tangential derivative on the characteristic surface Ω, is

$$\tilde{\partial}_j = s_j^\alpha \frac{\partial}{\partial x_\alpha} \equiv \; l_i (B_{ij}^{(\alpha)} - \chi_\alpha A_{ij}) \frac{\partial}{\partial x_\alpha}. \tag{38}$$

- Since

$$n_\alpha s_j^\alpha = l_i A_{ij} (c - n_\alpha \chi_\alpha) = 0, \;\; \text{for each } j, \tag{39}$$

the derivative $\tilde{\partial}_j$ on u_j is a tangential derivative also on the *wavefronts* Ω_t.

Dynamics of a Hyperbolic System of 1*st* Order PDEs Conti. $\cdots$

- The n tangential derivatives $\tilde{\partial}_j$ $(j = 1, 2, \ldots, d)$ contains only spatial derivatives and can be expressed in terms of any $d-1$ of the d tangential derivatives L_α, defined in terms of $\frac{\partial}{\partial \eta^\alpha_\beta}$ in (31) by

$$L_\alpha = n_\beta \frac{\partial}{\partial \eta^\alpha_\beta}, \qquad \alpha = 1, 2, \ldots, d. \tag{40}$$

- The operator L_α can also be written in the form

$$L_\alpha = n_\beta \left(n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta} \right) = \frac{\partial}{\partial x_\alpha} - n_\alpha \left(n_\beta \frac{\partial}{\partial x_\beta} \right), \ i.e.$$

$$\boldsymbol{L} = \boldsymbol{\nabla} - \boldsymbol{n}\langle \boldsymbol{n}, \boldsymbol{\nabla} \rangle. \tag{41}$$

- Thus $\boldsymbol{L}$ is obtained from $\boldsymbol{\nabla}$ by subtracting from $\boldsymbol{\nabla}$ its normal component. Beautiful, only tangential component of the spatial gradient $\boldsymbol{\nabla}$ remains.

Bicharacteristic Theorem

Theorem: For the hyperbolic system (16) of n first order PDEs we have the following results.

The ray equations are (see (22) and (30))

$$\frac{dx_\alpha}{dt} = \frac{\boldsymbol{\ell} B^{(\alpha)} \boldsymbol{r}}{\boldsymbol{\ell} A \boldsymbol{r}} \equiv \chi_\alpha \qquad and \tag{42}$$

$$\frac{dn_\alpha}{dt} = -\frac{\ell}{\boldsymbol{\ell} A \boldsymbol{r}} \boldsymbol{\ell} \left\{ n_\beta \left(n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta_\beta^\alpha} - c \frac{\partial A}{\partial \eta_\beta^\alpha} \right) \right\} \boldsymbol{r} \equiv \psi_\alpha . \tag{43}$$

The transport equation along a ray is (see (36)

$$\boldsymbol{\ell} A \frac{d\boldsymbol{u}}{dt} + \boldsymbol{\ell}(B^{(\alpha)} - \chi_\alpha A) \frac{\partial \boldsymbol{u}}{\partial x_\alpha} + \boldsymbol{\ell} C = 0, \tag{44}$$

We have not only derived but, on the previous slides, also given the explanations and significance of the various terms in the three equations of this theorem.

A Few Comments

- In 1970 when I was working on the stability of transonic flows, I could find reference to the word bicharacteristic only in Courant and Hilbert, Vol II.
- C&H (1962) had only the equation (42) and called it "Lemma on Bicharacteristic Directions". It attracted my attention and I started using this word in my publications since 1973.
- Note a careful use of "Directions" in C&H, where Courant did not pay attention to diffraction of the ray due to inhomogeneities in the medium.
- My bicharacteristic theorem also includes
  (1) diffraction of the ray
  and
  (2) a transport equation for the amplitude of the wave along a bicharacteristic.

A Few Comments $\cdots$ Cont.

- C&H has both a geometrical and an algebraic proof of (40). But our algebraic proof is much simpler and is a part of the general theorem.
- Derivation of the full ray equations was given in my paper in 1975.
- Full form of the theorem was given in research monograph [3].
- The theorem not only has a significance as a mathematical result but in applications, see [4,5].

A Few Comments Conti. ...

The Bicharacteristic Theorem has led to the development of a very active subject of research "Evolution Galerkin Method".

Lukacov'a Medvidov'a M., Morton K. W., Warnecke G. *Evolution Galerkin methods for hyperbolic systems in two space dimensions*, Math. Comp., **69**:1355–1384, 2000.

There are very nice incidences in 1992, when the development of the subject started in Bangalore and Oxford.

A review article on this new area of research is available in

Lukacov'a Medvidov'a M., Morton K. W. *Finite Volume Evolution Galerkin Methods*, Indian J. Pure Appl. Math., **41**, 329-361, 2010.

[1] C. M. Dafermos, *Hyperbolic Conservation Physics Laws in Continuum Physics*, Spinger, 2016. This book is in 854 pages.

[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics, vol 2: Partial Differential Equations*, Interscience Publishers, 1962.

[3] Phoolan Prasad and Renuka Ravindran, *Partial Differential Equations*, Wiley Eastern Ltd and John Wiley & Sons, 1984.

[4] Phoolan Prasad, *Nonlinear Hyperbolic Waves in Multi-dimensions*. Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, **121**, 2001.

[5] Phoolan Prasad, *Propagation of a Curved Shock and Nonlinear Ray Theory*, Pitman Researches in Mathematics Series, **292**, Longman, Essex, (1993).

[6] Phoolan Prasad, *Propagation of Multi-Dimensional Nonlinear Waves and Kinematical Conservation Laws*, Springer, Springer Nature Singapore, 2018, DOI 10.1007/s12044-016-0275-6

Thank You!