A PEEP INTO FOUR DIMENSIONAL SPACE

PHOOLAN PRASAD

Abstract. Four dimensional space (4-D space) hold a great mystery to senior students in schools and also to common people who have some interest in physics and mathematics. They have all heard of the Albert Einstein' s relativity and they think that 4-D space exists geometrically, only they are unable to comprehend it. Actually 4-D space is purely a mathematical idea and has no physical reality. This article is aimed at school level students and general public with examples. We also show article we show beautiful projections of figures of 4-D objects.

The author was encouraged to prepare the first lecture on this topic for class 10-12 students in 1989 after reading a chapter in *Mathematical Carnival*, Penguin Books, 1975 by Martin Gardner. Since then he has been been modifying his lectures and found that students were very thrilled and responded with many questions. The last lecture was at the Indian Academy of Sciences on 13 June, 2018. This article is based on the last lecture but follows the pattern of the lecture, which makes it easier for the students to understand. I skip here a discussion of relativity.

1. Introduction

List of Some Frequently Used Symbols or the Abbreviations:

n-D: n-dimesionalspace, STR: Special Theory of Relativity

$\mathbb{R}$ = set of real numbers = set of rationals + set of irrational, it also denotes the real line

$\mathbb{R}^n = (x_1, x_2, \cdots, x_n)$, where $x_i \in \mathbb{R},\ i = 1, 2, \cdots, n$ and $\Big(.,\ \ .,\ \ .\Big)$ is symbol in which the n entries have definite order. It is a vector with n components

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1.1. **Dimension of a Space Through Examples:**

- **1.** Zero dimensional space: A point.
- **2.** One dimensional space: A straight line.
- **3.** A train moves in one dimensional space - forward or backward.
- **4.** Two dimensional space: A plane. A ship on the surface of ocean moves in two dimension space.
- **5.** Three dimensional space: An aircraft flies in 3-dimensional space.
- **6.** Where is an example of a four dimensional space?
- **7.** What is your answer? -.-.-.-. I do not know? or -.-.-.-

1.2. **Dimension of a Space Through Examples - continued:**

- A mathematician peeps into it with his imagination and that is what we shall do in this article.

1.3. **4-Dimensional Space of relativity:**

- **1.** A great revolution took place in 1905 with discovery of the theory of relativity by Albert Einstein. Before that, time and space were assumed to be independent.
- **2.** Einstein explained that for persons in relative motion "there is no absolute time, and space and time get mingled up".
- **3.** Physicists use phrase "four dimensional space: (x, y, z, t)" of relativity or simply space-time. It is a mathematical idea.
- **4.** Due to this people, who do not understand SRT, think that 4- dimensional space exists geometrically, only they are unable to comprehend it.
- **5.** I would like to make it clear that our topic "four dimensional space" is purely a mathematical idea and has no physical reality.
- **6.** The *space-time* of the theory of relativity only means that the real 3-D space gets mingled with time.
- **7.** For a mathematician, it is a simple matter to define these concepts.
- **8.** Before that we need to see the relation between real line and the set of real numbers.

2. Relation Between Real line and the Set of Real Numbers and n-dimensional Space

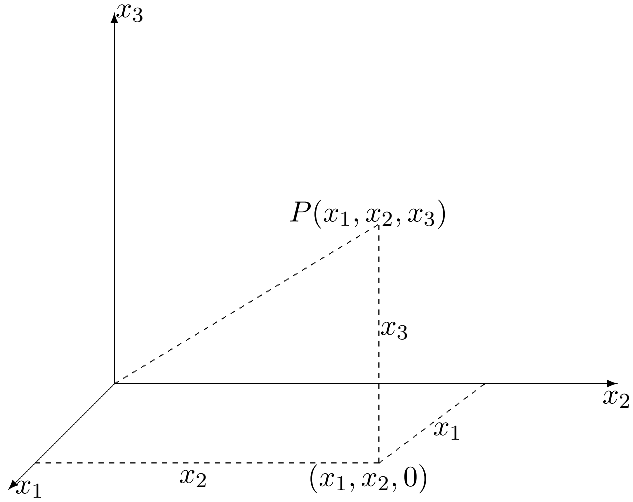
2.1. **Popular idea:** **1.** 0-dimensional space - a point. **2.** 1-dimensional space - a line. **3.** 2-dimensional space - a plane. **4.** 3-dimensional space - space where we live. **5.** 4-dimensional space - we do not know.

2.2. **Real number system and a line:** **1.** Notation $\mathbb{R}$ = set of real numbers = set of rationals + set of irrationals. **2.** Notation: $\mathbb{R}^n = (x_1, x_2, \cdots, x_n)$, where $x_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$. **3.** Notation $\big(., ., .\big)$ is symbol in which the entries have definite order. It is a vector with n components. **4.** There is one to one correspondence between points on a line and $\mathbb{R}$. Therefore we denote one dimensional space, i.e., a line by $\mathbb{R}$. Note this convenient notation.

2.3. **Mathematician's definition of n-dimensional space:** Continuing from the previous section **1.** A point in a plane is represented by an ordered pair of real numbers, i.e., thus 2-dimensional space is denoted by $\mathbb{R}^2$. **2.** A points in space is represented by an ordered triple of real numbers. **See figure 1.** Thus 3-dimensional space is denoted by $\mathbb{R}^3$.
3. $\mathbb{R}^4$ is a 4-dimensional space. Note here a change in sequence of words.

4. . . . . **5.** $\mathbb{R}^n$ is a n-dimensional space.

Mathematicians definition of n-dimensional space is simple and elegant, and without any ambiguity.



One to one correspondence between $\mathbb{R}^3$ and 3-D space. Geometrical visualization of components of coordinates for $\mathbb{R}^n$ is possible as shown above. See figure 13 later.

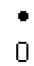
2.4. **Coordinates in 3-dimensional space:**

2.5. **Examples from mathematics: 1.** Consider the set of all polynomials of degree 2 or less with real coefficients, i.e. $a_0 + a_1x + a_2x^2$. **2.** In order to get a deeper understanding of the structure of this set, a mathematician formulates this as: **3.** Consider the space of all polynomials of degree 2 or less with real coefficients. **4.** Then he asks: What is the dimension of the space of all polynomials of degree 2 or less with real coefficients. **5.** Is it a meaningful or meaningless question? Can we call any set as "space" and ask for its dimension?

We put the matter in a slightly different way: 1. Given a polynomial $a_0 + a_1x + a_2x^2$, we get a triplet (a_0, a_1, a_2). **2.** Given a triplet (a_0, a_1, a_2) we can construct a polynomial $a_0 + a_1x + a_2x^2$. **3.** The space of polynomials of degree 2 is in one to one correspondence with the the space of ordered triplets, i.e. $\mathbb{R}^3$. Thus, space of all polynomials of degree 2 with real coefficients: $a_0 + a_1x + a_2x^2$ is 3-dimensional.

Without any ambiguity we state 1. When $a_1 = 0, a_2 = 0$, we have a plynomial of degree 0, which corresponds to a set (a_0) of a single real number, i.e. the x_1-axis, which is $\mathbb{R}$. **2.** When $a_2 = 0$, we have a polynomial of degree 1 or less, which corresponds to ordered pair (a_0, a_1) i.e. the space $\mathbb{R}^2$. **3.** A polynomial of degree 2, $a_2 \neq 0$, corresponds to a general point (a_1, a_2, a_3) in 3-D, i.e. $\mathbb{R}^3$. **4.** The space of all polynomials of degree n or less with real coefficients:

$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is $n + 1$ dimensional, i.e. $\mathbb{R}^{n+1}$.



1. A point $\{x_1 = 0\}$ on $\mathbb{R}$.

These are examples of mathematical spaces of finite dimensions. Now we give an example of an infinite dimensional space. We first define a Power Series

Definition 2.1. *Power Series: A power series (in one variable) is an infinite series of the form*
$\sum_{n=0}^{\infty} a_n (x-c)^n$
where a_n represents the coefficient of the nth term and c is a constant. a_n is independent of x and may be a function of n.

1. What is the dimension the space of a power series $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$?
2. The space of a power series $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ is same as the space of an infinite sequence $(a_0, a_1, a_2, \cdots, a_n, \cdots)$ which is infinite dimensional.

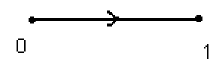
Remark 2.2. *4-dimensional relativistic space "space-time" is a mathematical concept. Relativity is difficult only when we try to visualize its physically realistic results geometrically, which certainly does not exist.*

3. Visualisation of a n-D Objects Geometrically

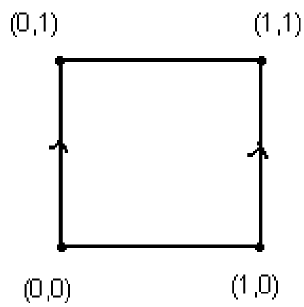
3.1. **Simple Examples from Mathematician's Point of View. 1.** A point $x = 0$ on the real line is zero-dimensional space. **2.** Real line is one-dimensional space $\mathbb{R}$.
3. But it is now important to note that:
3a. $0 \le x_1 \le 1$ is a segment of a straight line of unit length and is an one-dimensional object (not space). 3b. $x_1^2 + x_2^2 \le 1$ is a circular region in a plane (a part of two dimensional space) and is a 2-D object. 3c. $x_1^2 + x_2^2 + x_3^2 \le 1$ is a sphere in 3-dimensional space. 3d. $x_1^2+x_2^2+x_3^2+x_4^2 \le 1$ is a sphere in 4-dimensional space but we can not visualise it geometrically.

Mathematician need not attempt to visualise geometrically: 1. We may write for a 4-D object: modulus of the sum of the sequences of 4 real numbers is ≤ 1 or $|x_1+x_2+x_3+x_4| \le 1$ but without thinking seriously in terms of hyperspace figures. This is a four dimensional hyper cube, but it it harder for you to visualise. **2.** Start with the object $|x_1+x_2| \le 1$, count all faces of $|x_1+x_2+x_3| \le 1$, then go to $|x_1+x_2+x_3+x_4| \le 1$. This is an exercise for you.

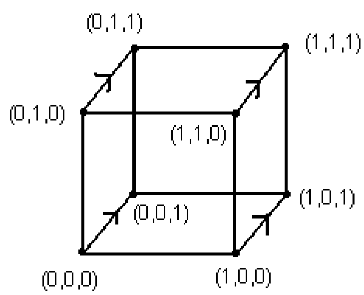
3.2. **Let us attempt to visualise n-D Objects (which are not spaces) geometrically:**



2. Move point by the unit distance along a straight line to generate line segment, $\{0 \le x_1 \le 1\}$ on $\mathbb{R}$.



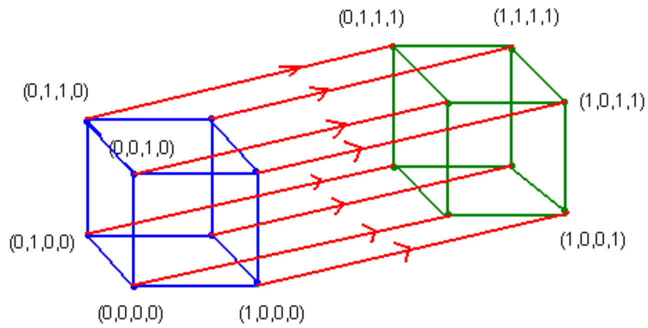
3. Move the line segment perpendicular to itself by the unit length to generates a square, $\{0 \le x_1 \le 1, 0 \le x_2 \le 1\}$ on $\mathbb{R}^2$.



4. Shift the square in a direction right angle to its plane i.e., in x_3 direction to get a cube $\{0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1\}$ on $\mathbb{R}^3$.

Our visual power ends there and we cannot proceed further. However, there is no logical reason why we can not assume that cube is shifted in a direction perpendicular to itself, i.e., in x_4 direction in (x_1, x_2, x_3, x_4)-space.

3.3. **Shifting a Cube in Fourth Direction to Get a Tesseract: 1.** If the cube is so shifted by unit distance, the object so generated is a unit hypercube, a tesseract in four dimensional space: $\{0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1, 0 \le x_4 \le 1\}$. **2.** Since there is no 4-D, we can not visualise it. But note that we drew a projection of the cube on a plane. **3.** Let us project a tesseract on 2-D and draw on the paper a sketch as shown in Figure 5.

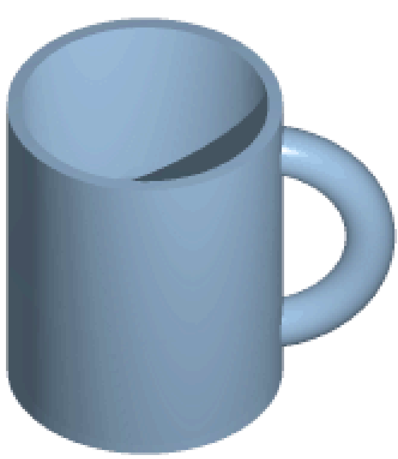


5. Tesseract is also called 8-cell or regular octachoron or cubic prism

4. Topologically same and different objects

- In 2-D, a circle and a square are topologically same. You can bend and you can stretch, but you cannot break and in this process you can deform a circle into a square.
- The circle is topologically different from a figure 8, because although you can squash the middle of a circle together to make it into a figure 8 continuously, when you try to undo it, you have to break the connection in the middle and this is discontinuous: points that are all near the center of the eight split into two batches and end up far apart on opposite sides of the circle.

Example of a topologically same objects from Wikipedia: A mug and a donut are topologically same.



6. A mug and a donut are topologically same.

Definition 4.1. *Two objects are topologically same if one of them can be continuously deformed by stretching and distorting into another and vice-versa.*

Topologically distinct objects: 1. We have drawn geometrical figures of a point, a line segment, a square and projections of a cube and tesseract in Euclidean spaces of dimensions 0, 1, 2, 3, 4 respectively. **2.** These objects are topologically distinct: a straight line can not be continuously deformed to a square*, a square deformed to a cube, a cube to a hypercube.

***Note:** Square means $\{0 \le x_1 \le 1, 0 \le x_2 \le 1\}$, not just the boundary.

Let us study some geometrical features of these objects, i.e. their number of corners, edges, number of squares and cubes:

Number of corners : **(1).** A unit line: 2 end points. **(2).** A square: $2 \times 2 = 4$ end points (corners). **(3).** A cube: $2 \times 4 = 8$ corners. **(4).** A tesseract: $2 \times 8 = 16$ corners.

Number of edges : **(1).** A unit line: 1 edge. **(2).** A square: movement of unit line - 2 from starting and end - 2 from the lines created by the corners (end points) i.e., $2 \times 1 + 2 = 4$ edges. **(3).** A cube: $2 \times 4 + 4 = 12$ edges (see figure 4). **(4).** A tesseract: $2 \times 12 + 8 = 32$ edges.

Number of Squares and Cubes: Number of squares :

(1). A square: 1 square. **(2).** A cube: by movement of a square. 1 starting square, 1 end square and 4 squares made by the 4 edges of square i.e., $2 \times 1 + 4 = 6$. **(3).** A tesseract: $2 \times 6 + 12 = 24$

Number of cubes : **(1).** A cube: 1 cube. **(2).** A tessaract: $2 \times 1 + 6 = 8$ cubes.

A Table for Corners, Edges, Squares and Cubes

n-spaces	Points	Lines	Square	Cubes	Tesseracts
0	1	0	0	0	0
1	2	1	0	0	0
2	4	4	1	0	0
4	16	32	24	8	1
5	...	...	...	...	...

Formula for n space: Expand $(2x+1)^n$. The coefficients of powers of x give the number of elements.For example, for $n = 4$, $(2x+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$.

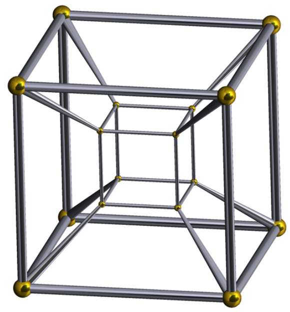
This is an algorithm, a proof is required.

5. Projections of the above objects

Projections of a teserract on a 3-D from Wikipedia, see figure 7:

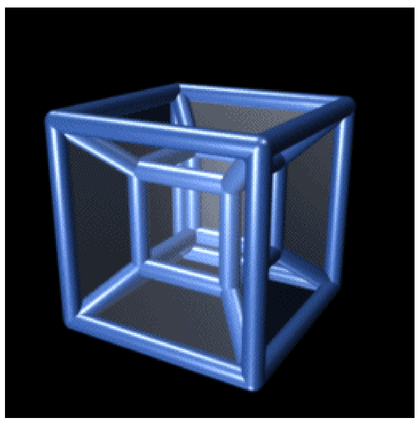
1. All elements of the tesseract can be identified. **2.** We have 8 cubes, six cubes suffer projective distortions. We have 6 hexahedrons surrounding the small cube.

Projections of a teserract on a 3-D from Wikipedia



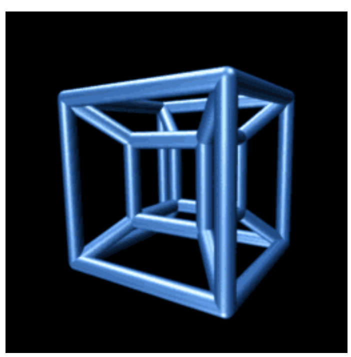
7. Projections of a teserract on a 3-D

3D projection of a tesseract performing a simple rotation



8. A 3D projection of a tesseract performing a simple rotation. Just enjoy the figure.

Double rotation of tesseract from Wikipedia

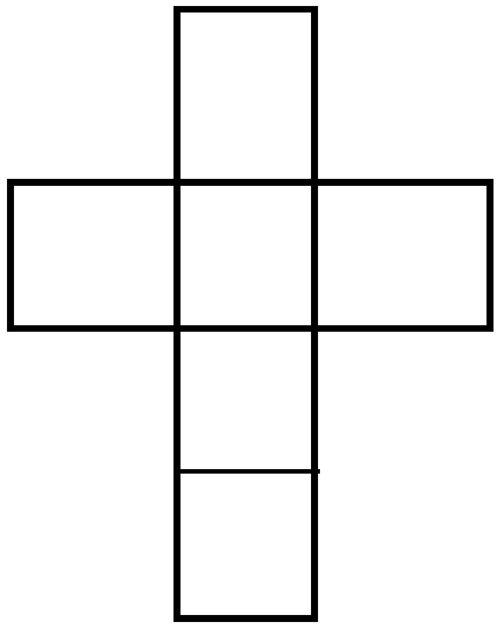


9. A 3D projection of a tesseract performing a double rotation. Just enjoy the figure.

6. Introduction Special Theory of Relativity (STR)

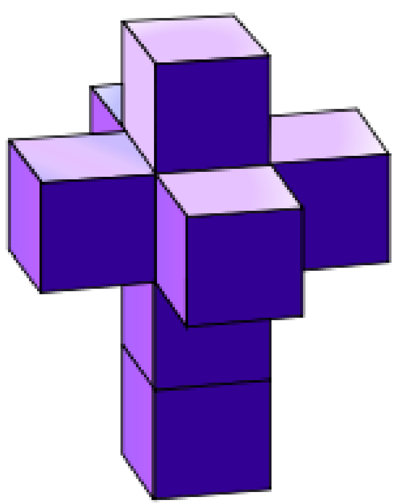
6.1. **Some questions: 1.** Suppose you are running very fast with a vertical pole and with a stone tied at the top of the pole. **2.** Suppose the stone suddenly falls. Where will it fall?

Unfolding of cube resulting 6 Squares in 2-D:



10. Unfolding of cube resulting 6 Squares in 2-D

Opening or Unfolding of Tesseract Resulting 8 Cubes in 3-D:



11. Unfolding of Tesseract from Wikipedia

3. Ptolemy (100–170 AD) says it will fall behind you. **4.** Galileo (1564–1642) says it will fall at your feet. **5.** Who is right?

6.2. **Hints for the answers:** **1.** Suppose you are travelling in a fast moving train and you drop a stone from your your hand. Where will it fall? **2.** What is your answer? **3.** What will be the path seen by a person on the ground outside the train? **4.** Problem for a reader - Draw trajectory in both frames.

Two important definitions:

- Position P: In a coordinate system in a space is represented by (x, y, z).
- Event: The stone occupies different positions at different time. We can associate an event (x, y, z, t) at a point of the trajectory.

6.3. **Introduction STR - Galilean transformation:** **1.** Suppose you have two frames S and S'. **2.** Let S' moves with a constant velocity $\boldsymbol{v}$ with respect to S. **3.** The direction of coordinate axes is at our disposal. We choose x-axis and x'-axis in the direction of the relative velocity. Now $\boldsymbol{v} = (v, 0, 0)$. **4.** Then our day to day experience of a common person gives the relation between the coordinates of frames S and S' the, Galilean transformation,

$$x' = x - vt,\ \ y' = y,\ \ z' = z,\ \ t' = t. \tag{6.1}$$

5. In what follows we take $v > 0$.

6.4. **Light trajectory in frame S for propagation in 1-D space:** **1.** Draw trajectory of light starting from $x = 0$ at time $t = 0$ for propagation in 1-D space with constant velocity c of light. **2.** We wish to draw in (x, t)-plane. **3.** The trajectory is

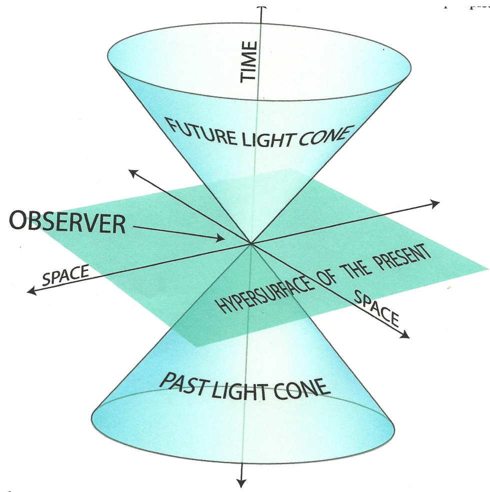
$$x - ct = 0 \ \textit{and}\ x + ct = 0 \qquad \textit{or both in one equation} \qquad x^2 - c^2t^2 = 0. \tag{6.2}$$

which we need to draw. Please draw it.

6.5. **Light cone in space-time with constant velocity of light c:** Let us draw the trajectory of light starting from origin at $t = 0$ in (x_1, x_2, x_3, t)-space, assuming velocity of light to be constant. **See figures 12 and 13.**

6.6. **Light cone in S' frame under the Galilean transformation:** **1.** The equations of above light cone in (x, y, z, t)-space and (x', y', z', t')-space have different expressions. **2.** First see it for 1-D propagation equation, (6.2) becomes $(x' + vt')^2 - c^2t'^2 = 0$. **3.** Therefore, light trajectory in S' is along two lines $(x' - (c - v)t) = 0$ and $(x' + (c - v)t) = 0$.
4. Unlike that in the frame S, these two lines are not symmetric with respect to the t'-axis.
5. In 3-D space, conoid becomes $x^2 + y^2 + z^2 - c^2t^2 = 0$ becomes $(x' + v + ct')(x' + vt' - ct) + y^2 + z^2 = 0$ and the t'-axis is no longer in the centre of the conoid.

Light cone in space-time with constant velocity of light c



12. Equation is $x^2+y^2+z^2-c^2t^2=0$. Interpret each part of this figure in 4-D: each section of the conoid by $t = constant$ plane is a sphere.

Light cone with constant velocity of light c in $(x_1, x_2, \cdots, x_n, t)$-space

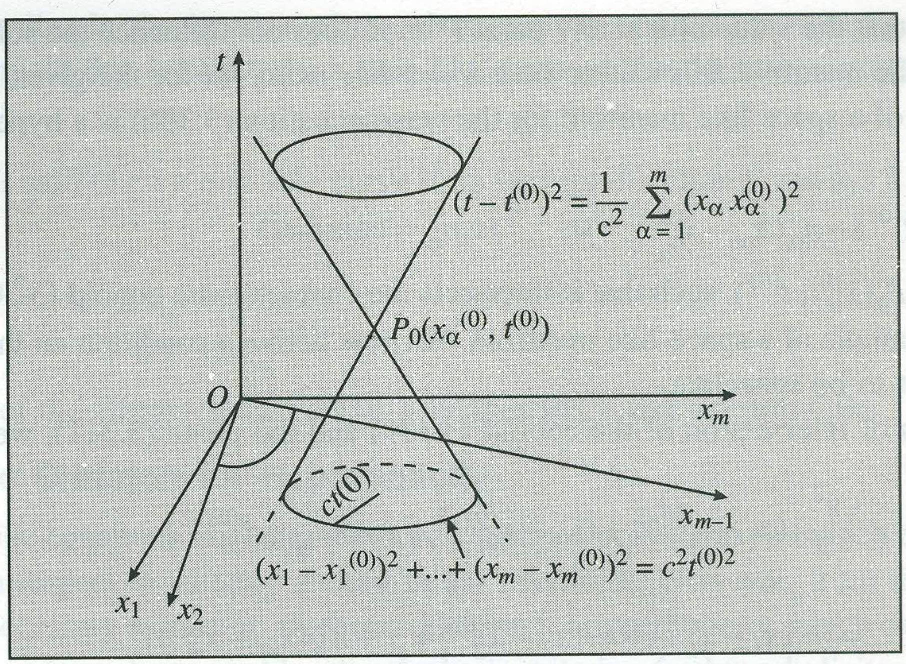


Fig. 3.21. *The lower and upper portions of the characteristic conoid through P_0 in space-time*

13. This is a generalisation in $n+1$-space dimensions.

6.7. **Einstein discovery:** Einstein found that Galilean transformation in section (6.6) is not valid but only approximately valid when the velocity $v << c$.

Inertial Frames: 1. Both, Newton and Einstein (Einstein for STR) talked about inertial frames. **2.** All inertial frames are in a state of constant, rectilinear motion with respect to one another. **3.** Mathematically, transformation from one inertial frame to another is given by a nonsingular linear transformation from (x, y, z, t) to (x', y', z', t'). Such transformation between (x, t) and (x', t') is:

$x' = a_1 x + a_2 t,\ t' = b_1 x + b_2 t$, with constants a_1, a_2, b_1 and b_2 satisfying $a_1 b_2 - a_2 b_1 \neq 0$.

Axioms of Special Theory of Relativity:

1. The two axioms of the *special theory of relativity* are (i) Laws of physics are same in all inertial frames. (ii) The speed of light in free space has the same value c in all inertial frames. **2.** Axiom (ii), required for derivation of the transformation giving STR can be stated as: *The light cone in space-time at* $(0, 0, 0, 0)$ *is the same for all inertial frames.* **3.** We have seen in section (6.6) this is not so with Galilean transformation.

STR - Derivation Lorentz transformation: 1. STR requires a linear transformation which comes from invariance of the light cone:

$x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$

2. Assuming that velocity of the second $\boldsymbol{v}$ is in the direction of the x-axis, from the above condition, we can derive Lorentz transformation, which we write below.

3. Lorentz Transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6.3}$$

$$y' = y \tag{6.4}$$

$$z' = z \tag{6.5}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6.6}$$

4. As a particular case, when $v << c$, we get approximately the Galilean transformation

$$x' = x - vt,\ y' = y,\ z' = z,\ t' = t.$$

Invariance of light conoid under Lorentz transformation - verification: Lorentz transformation gives

$$x'^2 - c^2 t'^2 = \frac{(x - vt)^2 - (t - \frac{v}{c^2}x)^2}{1 - \frac{v^2}{c^2}} \tag{6.7}$$

$$= \frac{x^2(1 - \frac{v^2}{c^2}) - c^2 t^2 (1 - \frac{v^2}{c^2})}{1 - \frac{v^2}{c^2}} \tag{6.8}$$

$$= x^2 - c^2 t^2. \tag{6.9}$$

Space-like plane and time-like direction:

In a space-like plane, (shown by R in the figure 14), in (x, y, z, t)-space, light signals from any point P on R do not reach any other point R. In the figure T, which cuts through the light cone, is not space-like.

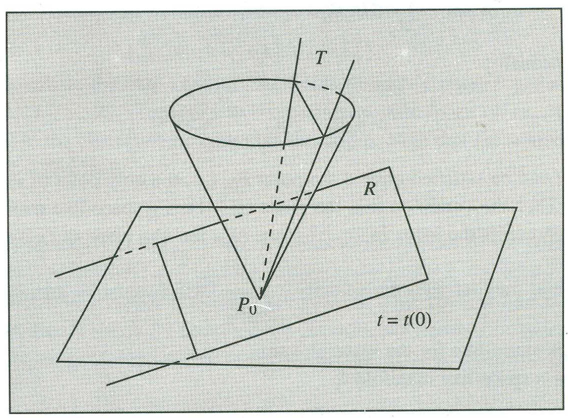


Fig. 3.22. *R is a space-like plane*

14. A time like direction points into the future light coniod or past light coniod (also called null cone)

We have shown at the end of the last page a beautiful result: Invariance in shape and position of conoid under under Lorentz transformation, when frames are in relative motion.

Final Comment:

1. I have shown you that there is no physical existence of 4-D space. **2.** But mathematicians do talk about n-D space and they can project figures on 2-D or 3-D, which you can visualize. **3.** The 4-D space of relativity is a mathematical space.

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Retired, Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

E-mail address: phoolan.prasad@gmail.com

URL: //math.iisc.ernet.in/~prasad/