The Bicharacteristic Theorem

(A theorem very useful for a deep understanding of solutions of hyperbolic systems of quasilinear equations and in developing numerical methods for their solutions)

Phoolan Prasad
Department of Mathematics
Indian Institute of Science, Bangalore - 560 012

Abstract

A light signal produced at time t_0 at a source $S((x_{10}, x_{20}, x_{30}) = \mathbf{x}_0)$ and moving with a point $P(\mathbf{x})$ reaches a point $P((x_{1t}, x_{2t}, x_{3t}) = \mathbf{x}_t)$ at a time t along a path, which is called a **ray**. This path is in the physical $\mathbf{x}$-space. A **bicharacteristic** is a curve in **space-time** $(\mathbf{x}, t)$, whose projection on the physical space (i.e. $\mathbf{x}$-space) is a ray.

The concept of bicharacteristic is not clear for wave propagation problems in 1-space dimension.

I first saw the word bicharacteristic in Courant and Hilbert, Methods of Mathematical Physics Vol-2 as "Lemma on Bicharacteristic Directions". It attracted my attention and I started using this word in my publications since 1973.

Note a careful use of "Directions" in C&H in the Lemma, where Courant did not pay attention to diffraction of the ray due to inhomogeneities in the medium. Later I formulated a full form of a bicharacteristic theorem, which included
(1) diffraction of the ray
and also
(2) a transport equation for the amplitude of the wave along a bicharacteristic.

I shall present this bicharacteristic theorem for a hyperbolic system of first order quasi-linear PDEs in multi-space dimensions ($\mathbf{x} = (x_1, x_2, \cdots x_m)$).

I had written the above abstract for a lecture on 27 March, 2021.

Additional Comments

(written on 8 May, 2021)

C&H has extensive reference to Hadamard but does not mention Hadamard for bicharacteristics. It is because of this that I did know Hadamard's renaming of Chauchy's characteristics as bicharacteristics for hyperbolic equations in muli-space dimensions.

I found a beautiful reference to bicharacterisstics by Luca Vitaglias "Characteristics, bicharacterisstics and geometric singularities of solutions of PDEs", 2014 at
*https://www.researchgate.net/publication/*258499449_*Characteristics* _Bicharacteristics _*and_Geometric_Singularities_of_Solutions_of_PDEs*
where he writs "wave fronts are characteristic surfaces and propagate along bicharacteristics". Vitaglia attributes bicharacteristics to Hadamard in the following sentence "Accordingly, characteristic surfaces are foliated by lines: characteristic lines in Cauchy terminology, bicharacteristic lines in Hadamard terminology. From a physical point of view, one concludes that a wave-front propagates along bicharacteristics."

C&H also discusses propagation of singularities in the solutions of hyperbolic equations along bicharacteristics and inspired by C&H, we (PP&RR book: *Partial Differential Equations*, John Wiley & Sons, 1984; reprinted by Wiley Eastern Ltd in 1985) give a coverage of this topic in sections 8.2, 8.3 and 8.4 in Chapter 3. This book is available freely on
http://www.math.iisc.ernet.in/~prasad/prasad/book/PP−RR_PDE_book_*1984.pdf*
All three research monographs of PP (1993, 2001 ans 2018) discuss bicharacteristics theorem in some detail with different proofs.

There are three aspects of a vector field on a manifold:

1. A fixed vector field on a manifold.
2. A vector field on a manifold which is driven by additional equations as it happens for a hyperbolic system of linear equations with coefficients depending on independent variables, you can see this in C&H. In this case the additional equations causes the vectors in the vector field change their directions with time.
3. A vector field on a manifold which is driven by a solution of the system in which the coefficients depend not only on independent variables but also on the dependent variables as in case of a hyperbolic system of quasi-linear equations. There are two sets of additional equations: (i) equations for the direction of the vectors, (ii) equations for the change of the direction of the vectors and another additional equation (iii) a compatibility condition for the solution. *An important point is that all the three sets of equations are **coupled**.*

Our bicharacteristic theorem contains all the attributes mentioned in point 3. The compatibility condition determines the solution along a bicharacteristic.

A review of the application of the theorem in formulation of numerical methods is available in: Lukacov'a Medvidov'a M., Morton K. W. *Finite Volume Evolution Galerkin Methods*, Indian J. Pure Appl. Math., **41**, 329-361, 2010. available at
*http://www.math.iisc.ernet.in/~prasad/prasad/book/*2010*~Mariia~Morton~INSA~paper.pdf*

Note: Two comprehensive and modern books, [1] Sylvie Benzoni-Gavage and Denis Serre, *Multidimensional Hyperbolic Partial Differential Equations: First-Order Systems and Applications*, Oxford University Press, 2007; [2] C. M. Dafermos, *Hyperbolic Conservation Physics Laws in Continuum Physics*, Spinger, 2016, by important mathematicians do not refer to bicharacteristics. Both Serre and Dafermos know me, have heard my lectures; and Dafermos even refers and classifies our work in differential geometry.

Essence of LUCA VITAGLIANO article by the author

In particular, I will describe in some details the transition (analogous to the transition from quantum mechanics to classical mechanics) focusing on intrinsic aspects, i.e., those aspects which are independent of the choice of coordinates. Differential geometry will be then the natural language.

The paper is divided into three sections. In the first section, I discuss Cauchy problems and characteristic Cauchy data. I conclude with some examples from Mathematical Physics. This section is CHARACTERISTICS AND SINGULARITIES OF SOLUTIONS OF PDES 3 basically analytic and makes use of local coordinates. However, most of the results therein are actually independent of the choice of coordinates. In the second section, I present the geometric setting for PDEs and their characteristics, specifically, jet spaces. Characteristics of PDEs have a nice, intrinsic definition in terms of jets. The geometric setting clarifies the relationship between characteristics and singularities of solutions. In the last section, I focus on **bicharacteristics**. Often characteristic surfaces are governed by a first order scalar PDE E. The geometry underlying such PDEs is contact geometry which is at the basis of the method of characteristics. It may happen that E is an Hamilton-Jacobi equation. There is a symplectic version of the method of characteristics for Hamilton-Jacobi equations based on the Hamilton-Jacobi theorem. This motives me to review the Hamilton-Jacobi theory. I conclude speculating about the possibility of extending the Hamilton-Jacobi theory to field theory in a covariant way, thus opening the road through a rigorous, covariant, Schrdinger quantization of gauge theories.

My colleague Gautam Bharali says that bicharacteristics appear in complex analysis in many complex variables. Probably, the origin may have been from the Garabedians work on complex characteristics starting from 1950 in an attempt to design shock free airfoils. I read this from the last chapter of Garabebian's book "PDEs" in 1967. There seems to extensive literature in this area in complex analysis and also in PDE in real variables, for example the work of Lars Hormander and later by others of the first decade of the 21 century.

I have noticed discussion on bicharacteristics in very complex systems of quasilinear equations (i) L. Hormander, Lectures on Nonlinear Hyperbolic Differential Equations, Springer, 1997 and (ii) C. Dappiaggi, V. Moretti and N. Pinamonti, Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime, International Press Adv. Theor. Math. Phys. 15 (2011) 355447 & - $e - print\ archive : http : //lanl.arXiv.org/abs/$0907.1034.

The topic "bicharacteristics" pervades in many areas of mathematics and physics.