

Nonlinear Fresnel diffraction of weak shock waves

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Fresnel diffraction at a straight edge is revisited for nonlinear acoustics. Considering the penumbra region as a diffraction boundary layer governed by the KZ equation and its associated jump relations for shocks, similarity laws are established for the diffraction of a step shock, an “N” wave, or a periodic sawtooth wave. Compared to the linear case described by the well-known Fresnel functions, it is shown that weak shock waves penetrate more deeply into the shadow zone than linear waves. The thickness of the penumbra increases as a power of the propagation distance, power 1 for a step shock, or $\frac{3}{4}$ for an N wave, as opposed to power $\frac{1}{2}$ for a periodic sawtooth wave or a linear wave. This is explained considering the frequency spectrum of the waveform and its nonlinear evolution along the propagation, and is confirmed by direct numerical simulations of the KZ equation. New formulas for the Rayleigh/Fresnel distance in the case of nonlinear diffraction of weak shock waves by a large, finite aperture are deduced from the present study. © 2003 Acoustical Society of America. [DOI: 10.1121/1.1610454]

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I. INTRODUCTION

Diffraction occurs in case of localized singularities of the geometrical approximation. These singularities can be of different types (Berry, 1981): infinite amplitude (caustics), discontinuity of the amplitude (shadow boundary produced by either a smooth or a sharp edge) or of the phase (so-called phase dislocations), and, for optical waves, polarization singularities. In the case of caustics, Berry (1976) has fruitfully related it to the mathematical theory of catastrophes (Thom, 1972).

Belonging to the class of amplitude discontinuities, one of the simplest and most studied singularities is Fresnel diffraction of a wave at a straight edge. The geometrical approximation predicts an abrupt discontinuity, the undiffracted wave on the illuminated side suddenly vanishing when entering the shadow zone. Diffraction effects take place around the shadow boundary, inside the *penumbra*, to match continuously the geometrical wave to the shadow zone. Following his 1816 memoir, awarded a prize by the Paris Science Academy in 1818, on diffraction fringes in the shadow zone, Fresnel, in subsequent works (collected in 1866), calculated the diffraction caused by straight edges, small apertures, and screens, and introduced the functions now bearing his name. For modern presentations, the reader is referred to the textbooks of Born and Wolf for optics (1975) or Pierce (1981) for acoustics.

The Fresnel function can also be recovered using the method of matched asymptotic expansions. Considering the penumbra region around the shadow boundary as a diffraction boundary layer, in this elongated region, the paraxial approximation of the wave equation describes the inner solution, which is matched to the undiffracted plane wave (the outer solution) far off the penumbra (Buchal and Keller, 1960; Zauderer, 1970). This presentation will be briefly re-

called in Sec. II, as it will be further used as a reference case to which nonlinear effects will be compared. The method of matched asymptotic expansion has also been successfully applied to other types of singularities. The description of the field around a fold caustic (the simplest of any caustic) by means of matched asymptotic expansions was introduced in the same article of Buchal and Keller (1960). This approach has since been extensively used, and numerous examples can be found in the books of Bouche and Molinet (1994), or Borovikov and Kinber (1994).

A key advantage of the matched asymptotic expansion is that it can also be extended to nonlinear acoustics. The nonlinear counterpart of the paraxial approximation is the well-known KZ equation (Zabolotskaya and Khokhlov, 1969). It is recalled in Sec. III, and completed by original derivation of the associated shock jump relations, as we are especially interested in weak shock waves here and neglect any absorption phenomenon which would have here only a negligible effect. The KZ equation was introduced initially to model finite-amplitude bounded beams (for a review, refer to Hamilton, 1998). A main advantage of the paraxial approximation is the subsequent one-way assumption, allowing one to solve the KZ equation numerically by advancing plane by plane away from the source. Also, omitting diffraction, the KZ equation reduces to the inviscid Burgers' equation, a very well-known equation for which reliable numerical algorithms have been developed, as proved by comparisons with exact solutions of weak shock theory. The coupling of both advantages has been used for the design of efficient codes, either in the frequency (Frøysa *et al.*, 1993) or in the time domain (Lee and Hamilton, 1995). Finally, despite the paraxial approximation, several comparisons with experiments have shown the high precision of the KZ equation, which is able to reproduce the finest details of the field radiated by large emitters at the fundamental and harmonic fre-

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quencies, either the main axis oscillations or the sidelobes (Hamilton, 1998).

However, the scope of application of the KZ equation is much more general, as it can model many diffraction effects localized along singularities, such as the Fresnel diffraction (as shown in the present study), the tip of finite fold caustics (Marchiano, 2003), or cusped caustics (Cramer and Seebass, 1978; Coulouvrat, 2000). A generalized version of the KZ equation models nonlinear diffraction in the shadow zone of an upward-refracting atmosphere (Coulouvrat, 2002). Omitting the propagation term in this generalized equation, it reduces to the nonlinear Tricomi equation modeling the focusing of weak shock waves at fold caustics (Guiraud, 1965). Another key point is that nonlinear similarity laws can easily be deduced by this approach. The so-called Guiraud's similitude shows that the maximum amplitude of a step shock focusing at a fold caustic varies as the power 4/5 of the amplitude of the incoming wave, obviously a nonlinear law. The same power law can be demonstrated at the tip of a finite fold caustic. For a cusp caustic, there is a similar law (Coulouvrat, 2000), which deviates even more from the linear law with a power 2/3. Therefore, there are numerous examples where intimate coupling between diffraction and nonlinearities occurs.

The objective of the present study is to provide a new example of such a coupling, devoted to nonlinear penumbra diffraction around a shadow boundary. The ingredients of the study will be the same as those described above: the nonlinear paraxial KZ equation and similitude rules for simple wave profiles. Compared to caustics, however, the singularity of the geometrical approximation is smoother here, the amplitude being discontinuous instead of infinite. Therefore, the coupling will be different too. For caustics, nonlinear effects play a key role in limiting the amplitude of the focused shock. For straight-edge diffraction, nonlinearities will be shown to modify the spatial extent of the diffracted zone (penumbra). Indeed, it will be demonstrated (Sec. IV) that the way a weak shock wave is diffracted strongly depends on the overall time waveform. For cases of importance ("N" wave or step shock), the diffraction boundary layer around the shadow boundary thickens more rapidly for weak shock waves than for linear waves. In other words, and this is the key result of this study, diffracted shock waves penetrate deeper into the shadow zone. Revisiting Fresnel laws almost two centuries later, we could say that nonlinearities enhance diffraction. The degree of this enhancement can be related to the nonlinear evolution of the frequency spectrum of the undiffracted wave. This theoretical result, obtained without solving explicitly the KZ equation, will be nicely confirmed in Sec. V by direct numerical simulations. With this in view, a new time-domain algorithm solving the inviscid KZ equation has been designed, with special emphasis on nonlinear effects (linear diffraction being handled in the same way as by Lee and Hamilton, 1995) which are solved exactly using weak shock theory for the potential, as suggested by Hayes, Heafeli, and Kulsrud (1969) for sonic boom applications. Finally, new expressions for the Rayleigh/Fresnel distance of an aperture of finite size will be deduced from the results of the present study. In the future, this may elucidate some non-

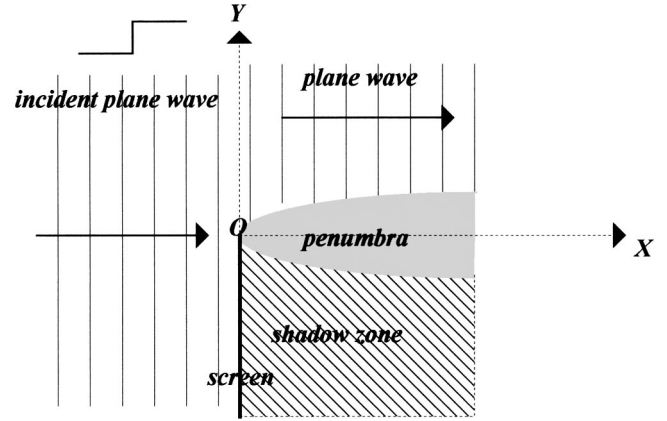


FIG. 1. Weak shock wave diffraction: Geometry of the problem.

linear behaviors of finite-amplitude beams, thus returning to the original application of the KZ equation.

II. LINEAR FRESNEL DIFFRACTION AT A STRAIGHT EDGE

In a two-dimensional, homogeneous, and inviscid fluid of ambient density ρ_0 and sound speed c_0 , a plane wave is incident normally on a perfectly thin, rigid, and semi-infinite screen. The Ox axis is oriented towards the direction of the incident wave, while the Oy axis is parallel to the screen, the origin being at its edge (Fig. 1). According to the geometrical acoustics approximation, the plane wave would propagate unaffected by diffraction in the upper half-space $y > 0$, while no acoustic field would exist in the shadow zone $y < 0$. The shadow boundary $y=0$ would be a singular line, the field amplitude being discontinuous there. Indeed, this is only a crude approximation: across this boundary, diffraction effects take place to match continuously the plane wave to the shadow zone. This *penumbra* region can easily be described by matched asymptotic expansions. As the penumbra (the inner region) is close to the shadow boundary, the paraxial approximation of the wave equation can be used at first order

$$\frac{\partial^2 P}{\partial X \partial \tau} = \frac{\partial^2 P}{\partial Y^2}, \quad (1)$$

where the dimensionless acoustic pressure is $P = p_a / \rho_0 c_0 U_0$, p_a being the acoustic pressure and U_0 the velocity amplitude of the incident field. The dimensionless retarded time is denoted $\tau = \omega(t - x/c_0)$, ω being the wave pulsation. The transverse variable y is scaled by a (yet unspecified) length $L: Y = y/L$, and the longitudinal one by the associated Fresnel distance $X = x/D$, with $D = 2\omega L^2/c_0$. For a pure monochromatic wave, one has $P(X, Y, \tau) = \hat{P}(X, Y) \exp(-i\tau)$, so that

$$-i \frac{\partial \hat{P}}{\partial X} = \frac{\partial^2 \hat{P}}{\partial Y^2}. \quad (2)$$

Indeed, the problem does not depend on any transverse scale, so the solution should be independent from the arbitrary quantity L . Therefore, the pressure field should be sought as a function of $\eta = Y/\sqrt{X}$, a combination independent of L . This yields the following equation:

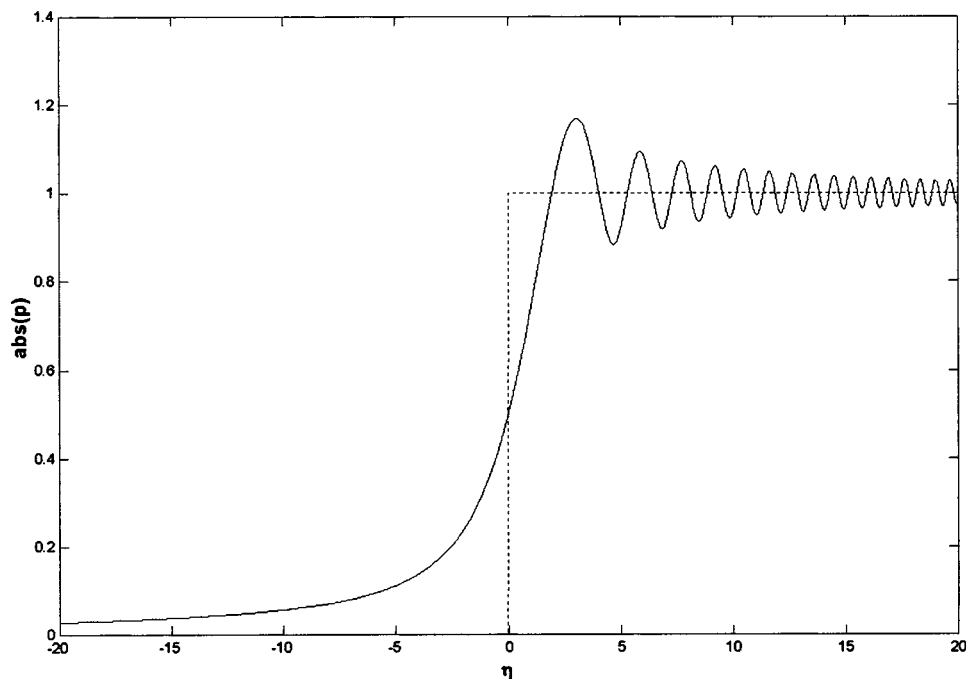


FIG. 2. Fresnel self-similar linear solution (continuous line) and geometrical approximation (dotted line).

$$i\eta \frac{d\hat{P}}{d\eta} = 2 \frac{d^2\hat{P}}{d\eta^2}, \quad (3)$$

with the boundary conditions

$$\hat{P}(\eta \rightarrow +\infty) = 1, \quad (4)$$

matching the field far from the shadow boundary to the transmitted plane wave unaffected by diffraction (the geometrical approximation, or in terms of matched asymptotic expansion, the outer expansion). The boundary condition in the shadow zone is simply

$$\hat{P}(\eta \rightarrow -\infty) = 0. \quad (5)$$

The solution of Eq. (3) is expressed in terms of the Fresnel integrals

$$\hat{P} = a \left[C\left(\frac{\eta}{\sqrt{2\pi}}\right) + iS\left(\frac{\eta}{\sqrt{2\pi}}\right) \right] + b, \quad (6)$$

where $C(x) = \int_0^x \cos(\pi t^2/2) dt$ and $S(x) = \int_0^x \sin(\pi t^2/2) dt$. The two constants a and b are determined according to the boundary conditions Eqs. (4) and (5), leading to the self-similar solution

$$\hat{P} = \frac{1}{2} \left(1 + (1-i) \left[C\left(\frac{\eta}{\sqrt{2\pi}}\right) + iS\left(\frac{\eta}{\sqrt{2\pi}}\right) \right] \right). \quad (7)$$

The fact that the pressure field is self-similar and depends on the variable η only, illustrates the existence of a *diffraction boundary layer* around the shadow boundary, whose thickness grows as the square root of the distance from the tip of the edge. Right on the singular line ($\eta=0$) $\hat{P}=1/2$, the pressure field is simply half the incident wave. Deep inside the shadow zone ($\eta \rightarrow -\infty$), one has

$$\hat{P} \approx \frac{-1}{\eta\sqrt{\pi}} \exp\left[i\left(\frac{\pi + \eta^2}{4}\right) \right];$$

the shadow field emanates from the edge wave diffracted at the tip of the screen and decaying as a cylindrical wave according to Huygens principle. This well-known Fresnel solution Eq. (7) is illustrated in Fig. 2, showing the characteristic diffraction fringes near the shadow boundary observed by Fresnel himself.

III. THE KZ EQUATION AND THE ASSOCIATED SHOCK RELATIONS

In nonlinear acoustics, the paraxial approximation of the wave equation is the well-known KZ equation (Zabolotskaya and Khokhlov, 1969)

$$\frac{\partial^2 P}{\partial X \partial \tau} - \frac{\partial^2 P}{\partial Y^2} = \frac{\partial^2}{\partial \tau^2} \left(\frac{P^2}{2} \right). \quad (8)$$

Contrary to the linear case, in the nonlinear case there does exist a physical scale D for propagation along the acoustical axis, which is the shock formation distance $D = 1/\beta k M$, with $\beta = 1 + B/2A$, B/A being the nonlinearity parameter and $M = U_0/c_0$ the acoustical Mach number. The corresponding transverse scale L is therefore given by: $L = 1/k\sqrt{2\beta M}$.

In the nonlinear, inviscid case, shock waves are likely to happen. Therefore, the KZ propagation equation is insufficient to solve the problem and must be supplemented by shock jump relations. These relations could be deduced from the general Rankine–Hugoniot shock relations (see, for instance, Landau and Lifshitz, 1959), applying on them the same asymptotic process enabling one to derive the KZ equation from the Euler fluid equations. However, a more straightforward derivation is proposed below, following the method of Whitham (1974). Indeed, an equivalent weak formulation of a balance law $(\partial a/\partial t) + \text{div } \mathbf{b} = 0$ (where generally, but not necessarily, t is time and \mathbf{x} is space) compatible with shock waves is obtained by integrating it over a volume V of surface S : $\partial/\partial t \int \int \int_V a(\mathbf{x}, t) d\mathbf{x} = \int \int_S \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{n} dS$. Now,

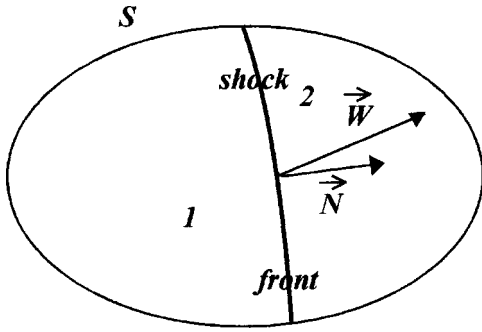


FIG. 3. Notations for shock jump relations.

assuming the existence of a shock wave located on surface Σ with unit normal vector \mathbf{N} oriented from side 1 to side 2 and moving at speed \mathbf{W} (Fig. 3), the usual jump relation satisfied across the shock wave can be proved

$$-W[a] = [\mathbf{b}] \cdot \mathbf{N}, \quad (9)$$

where $[f] = f_2 - f_1$ is the notation for the jump of any quantity f across the shock and $W = \mathbf{W} \cdot \mathbf{N}$ is the shock normal speed.

The KZ equation can be expressed in the form of two balance equations of the form below:

$$\frac{\partial P}{\partial X} = \frac{\partial U}{\partial Y} + \frac{\partial}{\partial \tau} \left(\frac{P^2}{2} \right), \quad (10a)$$

$$0 = \frac{\partial P}{\partial Y} - \frac{\partial U}{\partial \tau}, \quad (10b)$$

in which the variables (t, x, y) are identified with (X, Y, τ) , $a = P$, and $\mathbf{b} = (U, P^2/2)$ for the first conservation equation Eq. (10a), and $a = 0$ and $\mathbf{b} = (P, -U)$ for the second one, Eq. (10b). Physically, U is simply the (dimensionless) transverse acoustic velocity in the direction Y associated with diffraction effects, and Eq. (10b) is the usual linearized momentum equation in the transverse direction. Applying general jump relations (9) to Eqs. (10) yields

$$0 = [P]N_Y - [U]N_\tau \quad (11)$$

$$-W[P] = [U]N_Y + [P^2/2]N_\tau.$$

For the parabolic equation, the role of time is played by the propagation variable X . Therefore, the shock wave is considered as a line $Y_S(\tau, X)$ in the two-dimensional space (Y, τ) , with normal vector $\mathbf{N} = (N_Y, N_\tau)$, evolving with the propagation variable X . From Eqs. (11), it is possible to eliminate the jump of transverse velocity, so as to get

$$-WN_\tau = N_Y^2 + \langle P \rangle N_\tau^2, \quad (12)$$

where $\langle P \rangle = (P_1 + P_2)/2$ is the mean value of the pressure across the shock. In the one-dimensional case, $\mathbf{N} = (0, 1)$ and the jump relation Eq. (12) reduces to the well-known weak shock relation for the inviscid Burgers' equation $d\tau_S/dX = -\langle P \rangle$, or its equivalent geometrical form known as the "law of equal areas."

IV. SIMILITUDE LAWS FOR DIFFRACTED WEAK SHOCK WAVES

In a way similar to the linear self-similar Fresnel solution Eq. (7), we are looking for self-similar solutions of the KZ Eq. (8) and associated shock relation Eq. (12) in the case of an incident weak shock wave. Three cases are studied: a step shock, an "N" wave, and a periodic sawtooth wave. The geometrical approximation (the outer expansion) for these three cases would be on the illuminated side ($Y > 0$)

$$P(X, Y > 0, \tau) = \begin{cases} 0 & \tau < -X/2 \\ 1 & \tau > -X/2, \end{cases} \quad (13a)$$

$$P(X, Y > 0, \tau) = \begin{cases} -\tau/(1+X) & |\tau| < \sqrt{1+X} \\ 0 & \text{otherwise,} \end{cases} \quad (13b)$$

$$P(X, Y > 0, \tau) = -\tau/(1+X) \quad |\tau| < \pi \quad (13c)$$

(in the last equation, the pressure is periodic of period 2π), and 0 on the shadow side $Y < 0$. Sufficiently far from the edge of the screen ($X \gg 1$), the quantity $X+1$ can be identified with X in Eqs. (13b)–(13c). Undiffracting solutions (13) satisfy the inviscid Burgers equation and weak shock theory.

The inner solution in the penumbra must be solution of the KZ Eq. (8) and of the associated shock relation Eq. (12). Far from the penumbra $Y \rightarrow +\infty$, it must match the geometrical approximation Eq. (13) and vanish in the shadow zone $Y \rightarrow -\infty$. Searching for self-similar solutions, we introduce the following rescaling:

$$P \rightarrow P^* P, \quad X \rightarrow X^* X, \quad Y \rightarrow Y^* Y, \quad \text{and} \quad \tau \rightarrow \tau^* \tau, \quad (14)$$

the quantities $*$ denoting the rescaling amplitude of each corresponding variable. The rescaling (14) will leave the KZ equation invariant provided $P^* = \tau^*/X^*$ and $Y^* = \sqrt{X^*} \tau^*$. The invariance of the matching condition to the geometrical approximation implies the conditions:

$P^* = 1$ and $\tau^* = X^*$, and therefore $Y^* = X^*$ for the step shock Eq. (13a); $\tau^* = \sqrt{X^*}$ and $P^* = 1/\sqrt{X^*}$, and therefore $Y^* = X^{*3/4}$ for the N wave Eq. (13b); $\tau^* = 1$ and $P^* = 1/X^*$, and therefore $Y^* = \sqrt{X^*}$ for the sawtooth wave Eq. (13c).

Hence, the self-similar solutions must be sought in the form

$$P(X, Y, \tau) = Q \left(\eta = \frac{Y}{X}, \theta = \frac{\tau}{X} \right), \quad (15a)$$

$$P(X, Y, \tau) = \frac{1}{\sqrt{X}} Q \left(\eta = \frac{Y}{X^{3/4}}, \theta = \frac{\tau}{\sqrt{X}} \right), \quad (15b)$$

$$P(X, Y, \tau) = \frac{1}{X} Q \left(\eta = \frac{Y}{\sqrt{X}}, \theta = \tau \right), \quad (15c)$$

respectively for an incident step shock Eq. (15a), N wave Eq. (15b) and periodic sawtooth wave Eq. (15c). Substituting the above forms in the KZ equation, we get the following equations to be satisfied by the self-similar solutions, respectively, for the step shock Eq. (16a), the N wave Eq. (16b), and the periodic sawtooth wave Eq. (16c):

$$\frac{\partial Q}{\partial \theta} + \theta \frac{\partial^2 Q}{\partial \theta^2} + \eta \frac{\partial^2 Q}{\partial \eta \partial \theta} + \frac{\partial^2 Q}{\partial \eta^2} + \frac{1}{2} \frac{\partial^2 Q^2}{\partial \theta^2} = 0, \quad (16a)$$

$$\frac{\partial Q}{\partial \theta} + \frac{1}{2} \theta \frac{\partial^2 Q}{\partial \theta^2} + \frac{3}{4} \eta \frac{\partial^2 Q}{\partial \eta \partial \theta} + \frac{\partial^2 Q}{\partial \eta^2} + \frac{1}{2} \frac{\partial^2 Q^2}{\partial \theta^2} = 0, \quad (16b)$$

$$\frac{\partial Q}{\partial \theta} + \eta \frac{\partial^2 Q}{\partial \eta \partial \theta} + \frac{\partial^2 Q}{\partial \eta^2} + \frac{1}{2} \frac{\partial^2 Q^2}{\partial \theta^2} = 0, \quad (16c)$$

with the following boundary conditions to match the geometrical acoustics:

$$Q(\eta \rightarrow +\infty, \theta) = \begin{cases} 0 & \theta < -1/2 \\ 1 & \theta > -1/2, \end{cases} \quad (17a)$$

$$Q(\eta \rightarrow +\infty, \theta) = \begin{cases} -\theta & |\theta| < 1 \\ 0 & |\theta| > 1, \end{cases}$$

$$Q(\eta \rightarrow +\infty, \theta) = -\theta \quad |\theta| < \pi. \quad (17c)$$

In the shadow zone, the field vanishes in all cases

$$Q(\eta \rightarrow -\infty, \theta) = 0. \quad (18)$$

In time, for the step shock, the field is zero before the wave arrives (causality) and tends to a constant value after it has passed

$$Q(\eta, \theta \rightarrow -\infty) = 0, \quad \partial Q / \partial \theta(\eta, \theta \rightarrow +\infty) = 0. \quad (19a)$$

For the N wave, the field is equal to zero before the wave arrives (causality) and returns to zero after it has passed

$$Q(\eta, \theta \rightarrow \pm\infty) = 0. \quad (19b)$$

For the sawtooth wave, the signal is periodic

$$Q(\eta, \theta) = Q(\eta, \theta + 2\pi). \quad (19c)$$

Self-similar solutions must satisfy not only the KZ equation, but also the jump relation Eq. (12), as all solutions exhibit shock waves. This is checked by using the self-similar form Eqs. (15) and introducing the instant of shock $\theta_S(\eta)$ as a function of the self-similar distance across the diffraction boundary layer η . This yields the following equation of the shock line, respectively, for the step shock, the N wave, and the sawtooth wave:

$$(Y_S, \tau_S) = (\eta X, \theta_S(\eta) X), \quad (20a)$$

$$(Y_S, \tau_S) = (\eta X^{3/4}, \theta_S(\eta) X^{1/2}), \quad (20b)$$

$$(Y_S, \tau_S) = (\eta X^{1/2}, \theta_S(\eta)), \quad (20c)$$

where the variable η parametrizes the shock line at a fixed position X . From this comes the shock-front speed $\mathbf{W} = (\partial Y_S / \partial X, \partial \tau_S / \partial X)$, the shock-front tangent vector $\mathbf{T} = (\partial Y_S / \partial \eta, \partial \tau_S / \partial \eta)$, and the shock-front unit normal vector $\mathbf{N} = (-\partial \tau_S / \partial \eta, \partial Y_S / \partial \eta) / |\mathbf{T}|$. Substituting this into Eq. (12) yields the shock equation satisfied by $\theta_S(\eta)$ for the penumbra diffraction of, respectively, the step shock Eq. (21a), the N wave Eq. (21b), and the sawtooth wave Eq. (21c)

$$\theta_S(\eta) - \eta \frac{d\theta_S}{d\eta} + \left(\frac{d\theta_S}{d\eta} \right)^2 + \langle Q \rangle = 0, \quad (21a)$$

$$\frac{1}{2} \theta_S(\eta) - \frac{3}{4} \eta \frac{d\theta_S}{d\eta} + \left(\frac{d\theta_S}{d\eta} \right)^2 + \langle Q \rangle = 0, \quad (21b)$$

$$-\frac{1}{2} \eta \frac{d\theta_S}{d\eta} + \left(\frac{d\theta_S}{d\eta} \right)^2 + \langle Q \rangle = 0. \quad (21c)$$

These first-order differential equations must be completed by the boundary condition matching the diffracted shock position $\theta_S(\eta)$ to the undiffracted one as $\eta \rightarrow +\infty$, $\theta_S(+\infty) = -1/2$ for the step shock, $\theta_S(+\infty) = \pm 1$ for the N wave, and $\theta_S(+\infty) = \pi$ for the periodic sawtooth wave. This completes the set of Eqs. (15)–(19) satisfied by the self-similar solutions.

The main result of the present study is given by Eqs. (15). It demonstrates that the linear Fresnel laws of diffraction are substantially modified in nonlinear acoustics. Especially, it is proved that the diffraction of weak shock waves is strongly dependent on the shape of the waveform. While in linear acoustics, the diffraction boundary layer thickness (the penumbra) increases as the power $\frac{1}{2}$ of the distance from the diffracting edge, this result may not be true anymore in nonlinear acoustics. It remains so for the periodic sawtooth wave [Eq. (15c)], but the penumbra thickens as the power $\frac{3}{4}$ of the distance in case of an N wave [Eq. (15b)], and even linearly with the distance in case of a step shock [Eq. (15a)]. Hence, because of nonlinear effects, a diffracted step shock or N wave penetrates much deeper into the shadow zone than a linear or periodic sawtooth wave. For a step shock, that result could have been inferred easily from a basic dimension argument. Indeed, the dimensionless variables (X, Y) of the KZ equation both depend linearly on a typical wave number k . But, a step shock has no characteristic duration, and the problem should not depend on this arbitrary wave number. Therefore, it can depend only on the ratio Y/X , which is independent from k . The same reasoning explains the self-similar time variable τ/X , also independent from any frequency ω . For an undiffracting N wave, the duration of the signal increases (and therefore the frequency and the wave number decrease) with the square root of the distance. As, according to the linear Fresnel law, the penumbra thickness is proportional to $\sqrt{x/k}$, it becomes proportional to $x^{3/4}$ since k decreases with $x^{1/2}$: the diffraction boundary layer has to thicken faster than in the linear case to compensate for the nonlinear shift of the spectrum towards low frequency. On the contrary, the frequency spectrum of the periodic sawtooth wave remains constant, which explains why the penumbra thickness is similar to the linear Fresnel law in this case.

For more complex incident waveforms, the present study based on self-similarity cannot be used as such. However, significant effects are qualitatively expected. For instance, after nonlinear evolution, a finite pulse will be an intermediate between the N wave, similar to the start and the end of the signal, and a periodic sawtooth wave, for its main part. According to the above analysis, we can expect the start and end of the signal to penetrate deeper into the shadow zone, where they will be reinforced relatively to the main part so that the envelope of the pulse will flatten there. A more detailed analysis, likely to depend on the duration of the signal and the shape of its envelope, is beyond the scope

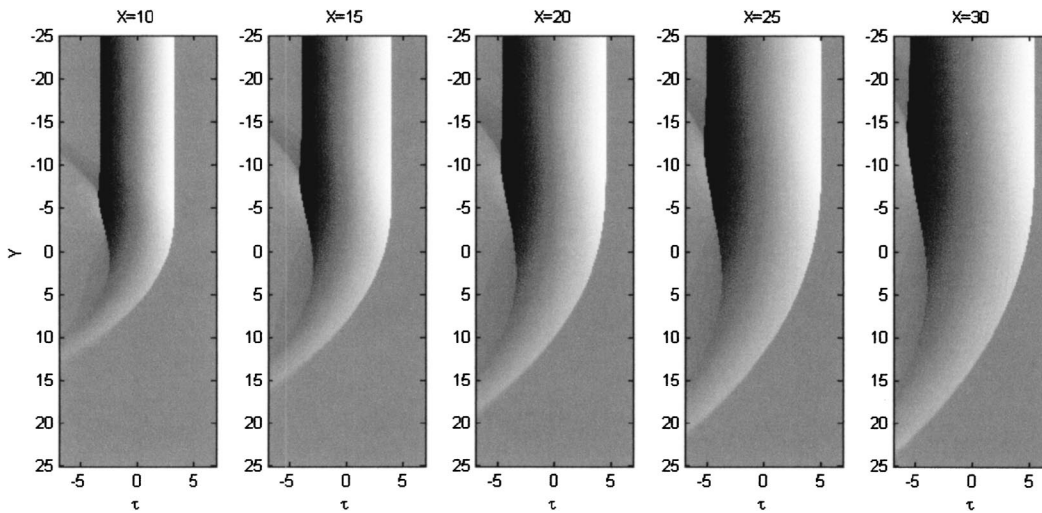


FIG. 4. Diffraction of an N wave. The penumbra solution of the nonlinear KZ equation is represented in physical dimensionless variables (horizontal: retarded time τ , vertical: transverse distance Y) at different distances X from the screen. The pressure level is represented in gray scales.

of the present analysis and would deserve further study relying on the numerical solver of the KZ equation presented below. Finally, despite its essentially theoretical objective, this study nevertheless demonstrates in practice that screens are likely to be less efficient against weak acoustical shock waves than could be expected from a purely linear analysis. Their design to protect from such type of loud sounds should necessarily take into account nonlinear effects.

V. NUMERICAL SIMULATION

To validate the theoretical results presented above, we compare them to a numerical simulation of the KZ equation Eq. (8). The incoming pressure field is specified in the plane of the screen $X=0$

$$P(X=0, Y, \tau) = \begin{cases} F(\tau) & Y > 0 \\ 0 & \text{otherwise,} \end{cases} \quad (22)$$

with the incident waveform

$$F(\tau) = \begin{cases} 0 & \tau < 0 \\ 1 & \tau > 0, \end{cases} \quad (23a)$$

$$F(\tau) = \begin{cases} -\tau & |\tau| < \sqrt{1} \\ 0 & \text{otherwise,} \end{cases} \quad (23b)$$

$$F(\tau) = -\tau \quad |\tau| < \pi, \quad (23c)$$

respectively, for a step shock Eq. (23a), an N wave Eq. (23b), and a periodic sawtooth wave Eq. (23c). For a numerical simulation, the KZ equation is expressed in terms of the potential $\Phi(X, Y, \tau) = \int_{-\infty}^{\tau} P(X, Y, u) du$

$$\frac{\partial^2 \Phi}{\partial X \partial \tau} - \frac{\partial^2 \Phi}{\partial Y^2} = \frac{1}{2} \frac{\partial}{\partial \tau} \left(\left(\frac{\partial \Phi}{\partial \tau} \right)^2 \right), \quad (24)$$

and is solved numerically advancing plane by plane away from the screen by small steps ΔX . For each step, the split-step method is used. At first, only diffraction effects are considered, and nonlinear ones on the right-hand side of Eq. (24) are omitted. This linear equation is discretized in the time

domain by finite differences according to the scheme of Lee and Hamilton (1995). This provides a first estimation of the solution in the new plane, taking into account only diffraction effects.

This estimation is improved by then omitting diffraction effects and recovering nonlinear ones, so that the KZ equation reduces to the inviscid Burgers' equation for the potential

$$\frac{\partial \Phi}{\partial X} = \frac{1}{2} \left(\frac{\partial \Phi}{\partial \tau} \right)^2. \quad (25)$$

An analytical solution of Eq. (25) is known, based on the implicit Poisson solution. In case of shocks, this solution may, however, be multivalued. Then, the physically admissible, single-valued solution is simply the maximum value of all multiple values, a test which is much simpler to implement from a numerical point of view than the weak shock theory, as in the Pestorius (1973) algorithm, or the law of equal areas. This procedure for solving the Burgers' equation via the potential was used by Hayes, Haefeli, and Kulsrud (1969) to determine sonic boom waveform distortion along acoustical rays launched by a supersonic aircraft. A full demonstration of the properties of the potential is given by Coulouvrat (2003). It has been applied to nonlinear diffraction problems in the case of shock wave focusing at fold caustics (Marchiano, Coulouvrat, and Grenon, 2003), where the pressure field satisfies the nonlinear Tricomi equation, which is a mixed-type transonic equation (elliptic/hyperbolic). Comparisons to experimental simulations (Marchiano, Thomas, and Coulouvrat, 2003) demonstrate the validity of this numerical approach.

Figures 4 and 5 present, respectively, the pressure field for a diffracted step shock or N wave at different distances X from the screen. The horizontal axis is τ , the vertical one is Y , the pressure levels are indicated by a gray scale. The wave diffraction in the penumbra, and its penetration inside the shadow zone is clearly observable in both cases. The self-similar aspect as expected from Eq. (15) is demonstrated by Figs. 6 and 7, where the *same* numerical solution is repre-

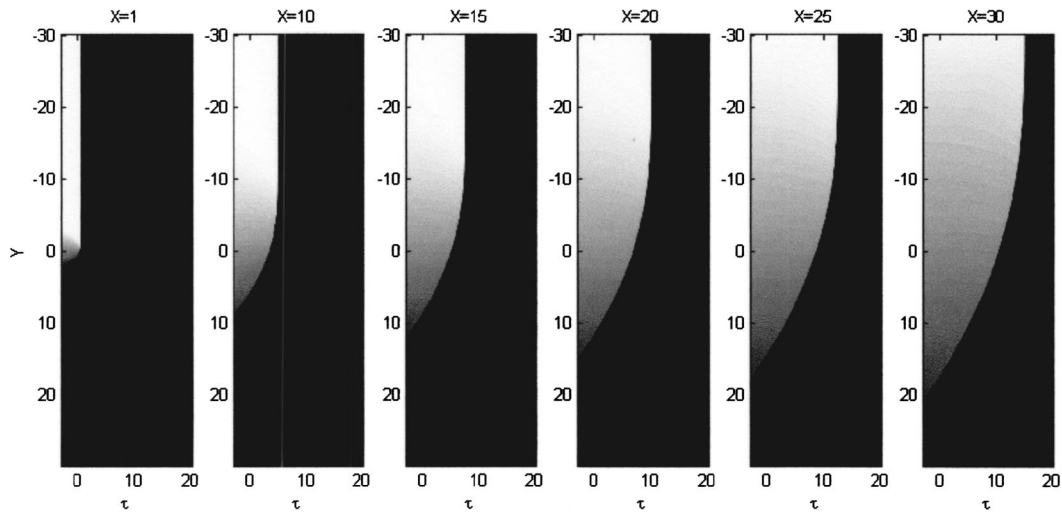


FIG. 5. Diffraction of a step shock. The penumbra solution of the nonlinear KZ equation is represented in physical dimensionless variables (horizontal: retarded time τ , vertical: transverse distance Y) at different distances X from the screen. The pressure level is represented in gray scales.

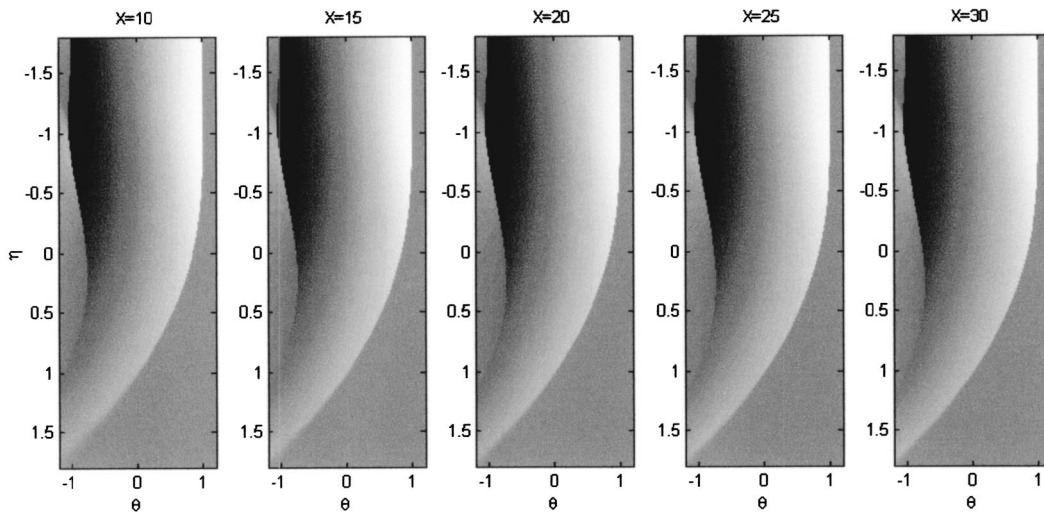


FIG. 6. Diffraction of an N wave. The penumbra solution of the nonlinear KZ equation is represented in self-similar variables (horizontal: θ , vertical: η) at different distances X from the screen. The pressure level is represented in gray scales. The invariance of the solution demonstrates the numerical solution is self-similar, in agreement with theory.

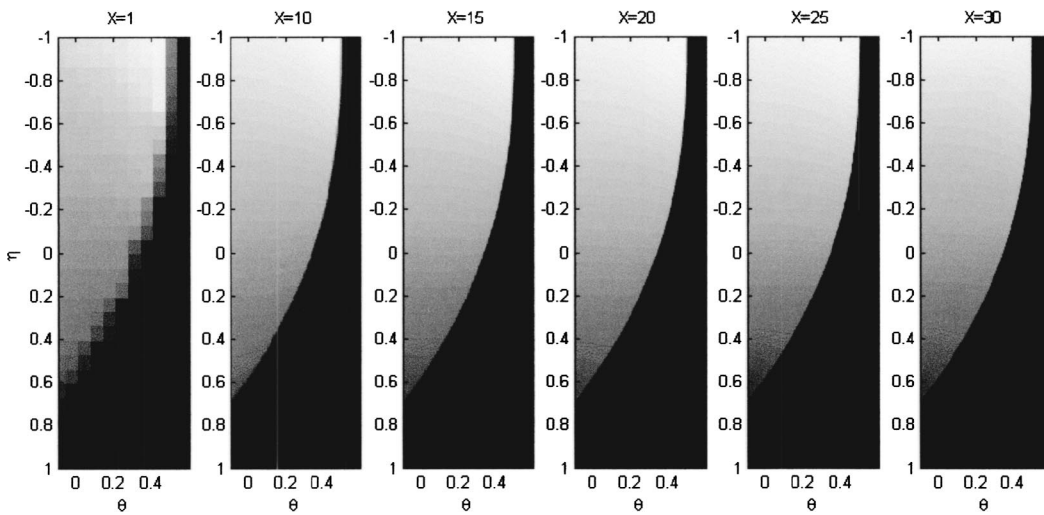


FIG. 7. Diffraction of step shock. The penumbra solution of the nonlinear KZ equation is represented in self-similar variables (horizontal: θ , vertical: η) at different distances X from the screen. The pressure level is represented in gray scales. The invariance of the solution demonstrates the numerical solution is self-similar, in agreement with theory.

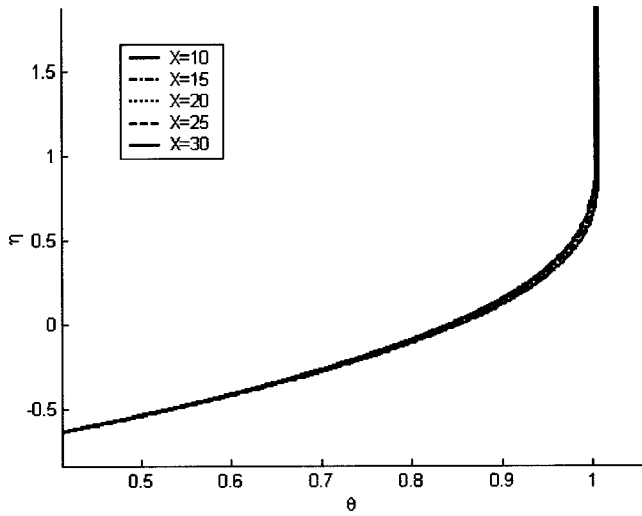


FIG. 8. Position of the first shock front of the diffracted N wave (numerical solution of the KZ equation) represented in self-similar variables (horizontal: θ , vertical: η) at different distances X from the screen. The superposition of the different curves demonstrates the numerical solution is self-similar, in agreement with theory.

sented in terms of the corresponding self-similar variables η and θ . In these variables, it is obvious that the numerical solution of the KZ equation is independent from the propagation distance. In other words, it is indeed self-similar, in agreement with the theoretical laws, Eq. (15). A more quantitative visualization of the self-similar aspect of the numerical solutions is illustrated by Figs. 8 and 9, showing the position of the initial shock front $\theta_s(\eta)$ reported as a function of the self-similar variables η and θ for different values of X . The almost-perfect superposition of the different curves proves that the numerical simulations of the KZ equation satisfy the nonlinear self-similar rules, Eq. (15).

VI. CONCLUSION

The theoretical analysis of Sec. III, confirmed by numerical simulations of Sec. IV, demonstrates that diffraction

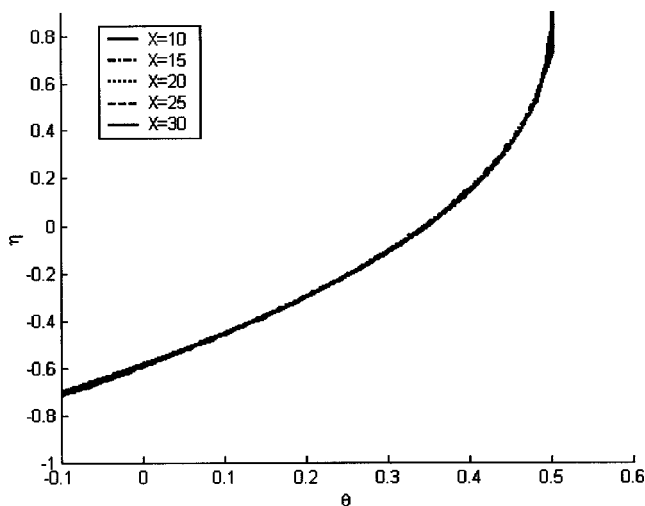


FIG. 9. Position of the shock front of the diffracted step shock (numerical solution of the KZ equation) represented in self-similar variables (horizontal: θ , vertical: η) at different distances X from the screen. The superposition of the different curves demonstrates the numerical solution is self-similar, in agreement with theory.

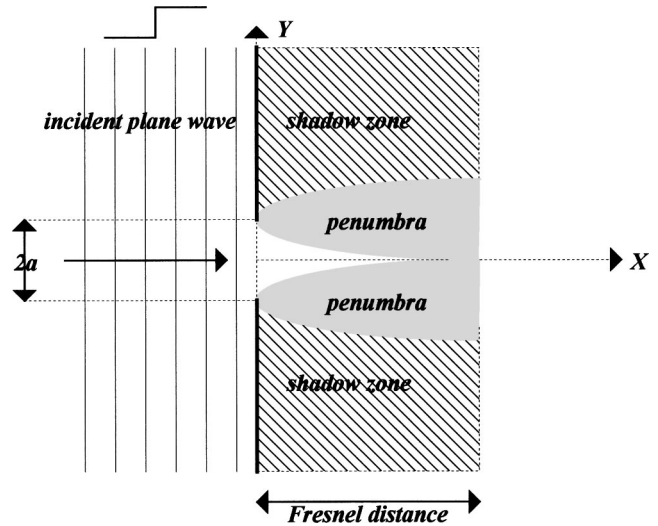


FIG. 10. Fresnel distance defined as the junction of the two penumbras emanating from the two edges of the slit.

of weak shock waves is intrinsically nonlinear and obeys similitude rules different from those of usual linear acoustics. Especially, it is proved that the thickness of the penumbra is dependent on the incident waveform, as it has to adapt to the evolution of the wave spectrum propagating nonlinearly far off the shadow boundary. Consequently, a step shock or an N wave penetrates deeper into the shadow zone than a periodic sawtooth wave: for weak shock waves, nonlinearities enhance diffraction.

An important consequence of this is a redefinition of the Fresnel/Rayleigh length for the case of diffraction by a slit of finite aperture $l=2a$ large compared to the wavelength. The Fresnel/Rayleigh length F can be estimated at the distance at which the two diffraction boundary layers emanating from each edge of the aperture do contact one another (Borovikov and Kinber, 1994) (Fig. 10). All points located beyond this distance will “feel” simultaneously both edge waves, which interfere in a transition between the near field and the far field. In linear acoustics, or for a periodic sawtooth wave, according to Fresnel self-similar solution, this gives $1 = \eta = y \sqrt{2k/x}$ for $y=a$ and $x=F$, so that $F=kl^2/2$. However, in nonlinear acoustics, the same reasoning would give $F = l/\sqrt{2\beta M}$ for the Fresnel/Rayleigh distance of a step shock, and $F=k^{1/3}l^{4/3}/(4\beta M)^{1/3}$ for an N wave. In the case of a step shock, it is of course independent from any frequency and depends only on the slit aperture (linearly) and wave amplitude, while for an N wave it involves an unexpected combination of frequency, aperture, and amplitude. For these last two cases, as diffraction is enhanced by nonlinearities, the Fresnel/Rayleigh distance varies as a smaller power with the aperture (power 1 or 4/3 instead of 2), so that the near field will be of lesser extent than in the linear case.

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