Title of Talk to Follow Later

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DEPARTMENT OF MATHEMATICS
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Science Academies Refresher Course

Partial Differential Equations and their Applications
July 03 - July 15, 2017

July 10, 2017

Report by John Scott-Russel, British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Approximate Shape of Wave Observed by Scott-Russell

First wave of translation.

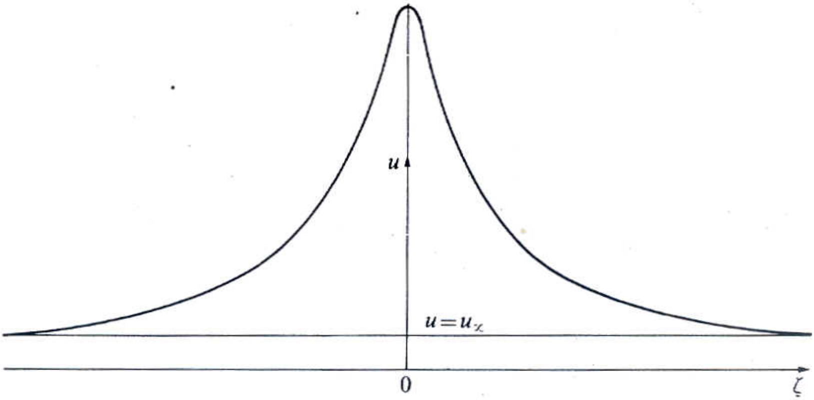
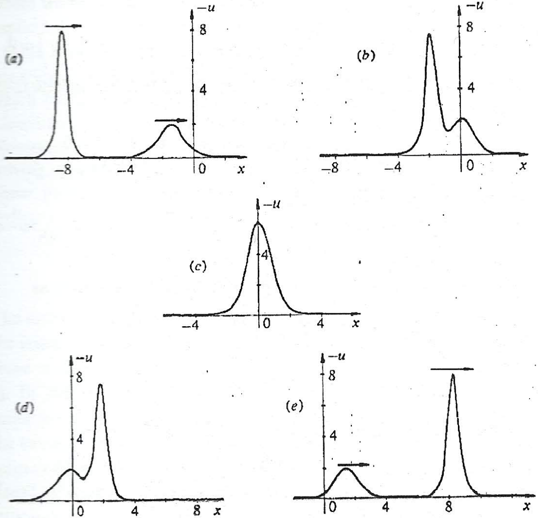


FIG. 2.7. Profile of a solitary wave joining constant states at $\xi = \pm\infty$ and localized in ξ.

Solitary Wave Became Soliton in 1967

Fig. 4.3 The two-soliton solution with $u(x,0) = -6\,\mathrm{sech}^2 x$ (see (c)); (a) $t = -0.5$; (b) $t = -0.1$; (d) $t = 0.1$; (e) $t = 0.5$. Note that $-u$ is plotted against x.



KdV Equation - Lecure 1
General Comments & Waves of Permanent Form

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My Choice of Talk

- I have chosen to talk on developments, which took place some 50 years back and is avilable in:
- "Nonlinear Waves in One-dimensional Dispersive Systems" BY PLB, carefully edited by me, OUP, 1979 and Drazin and Johnson, CUP, 1989.
- I have chosen this because
  1. Most of the teachers and students in mathematics departments in India have no idea of this topic and physicists here are interested only in computational aspects of solitons (not yet defined).
  2. Every PDE course covers "Laplace and diffusion equations", but sadly neglects "KdV equation", theory of which much deeper (shown in these lectures) and application wise at least as important.
  3. In my opinion we should cover this much neglected topic in every course on PDE

My Choice of Talk KdV E

- Kruskal writes in 1978, "KdV equation is arguably the simplest PDE equation $\cdots$ not covered by classical methods."
- In my opinion, it is also the most beautiful one.
- See Miles (1981) [4] for history, KdV E first appeared in thesis of de Vries (1894).
- KdV equation, in most explicit form showing all parameters, $\alpha, c\ and\ K > 0$, which are constants, is

$$u_t + cu_x + \alpha u u_x + K u_{xxx} = 0, \quad (1)$$

  c uniform velocity in unperturbed medium, α amplitude and K dispersion coefficient.
- Looks innocently simple.

KdV Equation

- Extensively quoted paper is by D. J. Korteweg & G. de Vries in 1895 [2]. They also obtained expression of above solitary wave and cnoidal wave, which we shall give later.

- Next significant contribution started coming from numerical work work of

  1. Fermi, Pasta & Ulam (1955) and
  2. Zabuski & Kruskal[1] (19665).

- Then, there was an explosion of research not just in KdV equation but many having same properties - P.D. Lax giving a new direction in 1968.

- I kept on watching.

[1] M. D. Kruskal was my guest at IISc

I Kept on Watching

- I was a research student since 1965 and kept on watching these beautiful developments since 1966.

- I desired to work on KdV equation but isolated in India, I had no mentor to show what to learn and where to begin.
- Subject, pursued by great physicists and mathematicians, was almost completed by 1974.
- But I had collected about 50 most important reprints, which formed material which PLB read and wrote first draft of book at MRI (no library there) and passed away in 1976.
- After returning to IISc I, with help of my some colleagues, edited the draft over a period of one year.
- See the foreword by Lighthill and my note.

KdV E derivation

- It first appeared as an approximate equation - governing surface water waves in which wave length is large compared to depth of water with dispersion included.
- Without dispersion it becomes Burgers' equation, solutions of which are waves coming towards a beach.
- Later it appeared as an approximation of a large number of quasilinear hyperbolic systems to which higher order dispersion terms are added.
- Let me not mention more, please see PLB [3].
- But I just mention important contribution of Boussinesq (1871, 1872 and 1877), for which refer to [4]. Use it to see local effects of other modes on solitons.
- A comprehensive discussion and extension of derivation of Boussinesq in 2-D is available in PP-RR (1977), [5].

Material a Compact Course or a Part of PDE Course

- It is very important to cover these developments in every graduate course in PDE, these are at least as much important as other classical equations - not just for mathematics but also for applications in many other sciences.
- Theory of KdV equation and associated developments is quite involved - concepts from physics are also mixed with KdV presentation.
- Hence we must do it at least with KdV equation first up to soliton interaction and second general equations of evolution by Lax.
- There is also an opportunity to present KdV as an abstract theory.

Suggestion for Material in a PDE Course

I shall suggest two topics for a course:

- KdV equation, inverse scattering method and derivation of solution of initial value problem leading one soliton and double soliton solution
  4 lectures.
- Introduction to a general nonlinear evolution equations as developed by Lax (1968) and as presented by Bhatnagar (1979) [3]
  2 to 3 lectures.
- In above topics give some brief background from physics but axiomatic presentation of theory is necessary in a mathematics course.
- Carefully separate where there is physics and where there is mathematics.

Aim

- Historically, first evolution equation for which Inverse Scattering method was developed is KdV Equation.
- I shall very briefly describe it in second lecture.
- Details are available in [1] and [3].
- Derivation of KdV Equation and other equations where solitons (not yet defined) is also available in many books, including above two books.

Simplest wave equation - A Part of the Wave Equation Waves - Second Wave of Translation

How do you solve it

$$u_t + c\, u_x = 0\ ,\ \ c = \text{real constant ?} \tag{2}$$

Method of characteristics for first order PDE gives

$$u = f(x - ct),\ \ f : \mathbb{R} \to \mathbb{R} \tag{3}$$

When $f \in C^1(\mathbb{R}) \Rightarrow$ Genuine solution

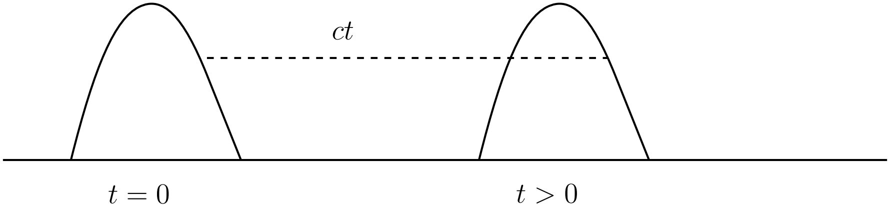


Figure: Wave of translation in which every point of pulse gets moved by the same distance ct, which is also true for KdV solitary wave. This figure does not reprent a genuine solution. Why?.

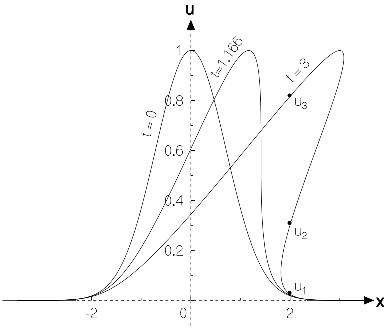
Nonlinear deformation due to genuine nonlinearity

- Airy (by Bona) or Euler (by Arnold) equation (common name Burgers equation)

$$u_t + uu_x = 0 \tag{4}$$

- This quasi-linear equation has *genuine nonlinearity*.
- Note that I have said it has, we shall see when we do hyperbolic equations.
- What do you get from its general solution $u = f(x - ut)$?
- $u|_t$ at x is same as what u was at $t = 0$ but $-ut$ behind it.
- Not easy to solve. why?

Nonlinear deformation due to genuine nonlinearity



The pulse now deforms (why?) as t increases, at $t = t_c = 1.166$ here the slope at a point becomes infinite and at $t = 3$, the graph does not represent any solution.

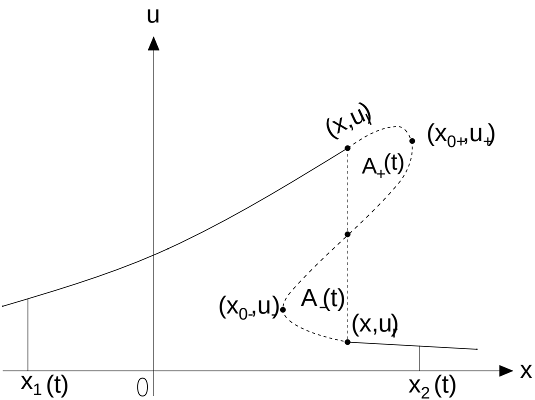
- When solution ceases to exist after some time, we need to change meaning of solution.
- For this we need a need a new mathematical formulation: original conservation law from which (4) is derived.
- One conservation law is (to be discussed in great detail by Arun)

$$u_t + (\frac{1}{2}u^2) = 0. \tag{5}$$

Burgers' equation can be derived from it if u is smooth.

- We shall have to look for its distributional solution, which may have discontinuities.
- Now a discontinuity appears at time $t > t_c$. This discontinuity is called shock - mathematical concept.

Nonlinear deformation contd..



When the graph folds at a large time, we need to interpret the solution as a weak solution with a discontinuity which is a shock. Discussion involves mathematical concepts which I do not pursue.

Simplest Solution with Shock

EXAMPLE 1

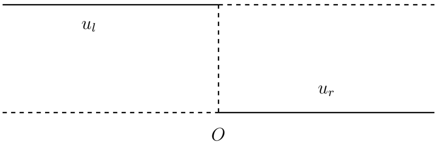


Figure: Initial data with a discontinuity at $x = 0$.

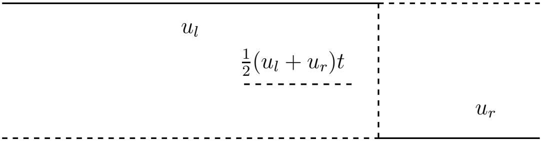


Figure: Solution, which is unique, remains discontinuous. The discontinuity moves with velocity $S = \frac{1}{2}(u_l + u_r)$.

Equation with diffusion and Diffusion with Genuine Nolinearity

Simplest example with diffusion is the heat conduction equation

$$u_t = \nu u_{xx}, \;\; \nu = \text{real and } > 0. \tag{6}$$

As time increases, concentration of u diffuses. Discussed in great detail by Baskar.

However, I shall mention here Burgers equation with viscosity (1948)

$$u_t + u\, u_x = \nu u_{xx} \tag{7}$$

In this both genuine nonlinearity and diffusion are present.

When diffusion balances accumulative effect of genuine nolinearity, we get third wave of translation.

Diffusion: Shock Structure. This is third Wave of Translation in my Presentation.

We look for solution $u = f(x - ct)$ - travelling wave of permanent form, with condition $u \to u_{\pm\infty}$ as $x \to \pm\infty$. Equation for f is ODE.

$$\begin{aligned} u(x,t) =& \frac{1}{2}(u_\infty^- + u_\infty^+) \\ & - \frac{1}{2}(u_\infty^- - u_\infty^+) \ \tanh \left[\frac{u_\infty^- - u_\infty^+}{4\nu} \left\{ x - \frac{1}{2}(u_\infty^+ + u_\infty^-)t \right\} \right] \end{aligned} \tag{8}$$

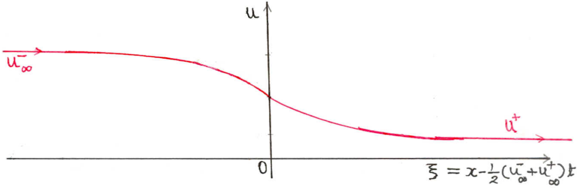


Figure: The genuine nonlinearity and diffusion balance each other and the discontinuous shock profile becomes a *steady* $C^\infty(\mathbb{R})$ solution.

Explain why $C^\infty(\mathbb{R})$.

Dispersion

Simplest example is

$$u_t = K u_{xxx} \ , \ K = \text{constant} \ > 0. \tag{9}$$

Substitution $u = e^{i(\omega t - kx)}$ gives, for $\omega =$ frequency, $k =$ wave number,

$$\text{dispersion relation } \omega = Kk^3 \tag{10}$$

$$\text{phase velocity } \frac{\omega}{k} = Kk^2 \tag{11}$$

$$\text{group velocity } \frac{d\omega}{dk} = 3Kk^2 \tag{12}$$

Initially, different frequency components tend to separate due to different phase speeds. Ultimately the wave moves with the group velocity.

In KdV equation dispersion balances effect of genuine nonlinearity, giving entirely new type of waves.

- We consider KdV E in the form

$$u_t + uu_x + Ku_{xxx} = 0, \qquad K > 0 \tag{13}$$

and look for its solution in the form

$$u(x,t) = h(\xi), \qquad \xi = x - ct. \tag{14}$$

- Above KdV E gives

$$-ch_\xi + hh_\xi + Kh_{\xi\xi\xi} = 0 \tag{15}$$

- Integrate twice with special care in second integration, we get with A and B as arbitrary constants

$$\left(\frac{dh}{d\xi}\right)^2 = \frac{1}{3K} f(h), \quad f(h) = -h^3 + 3ch^2 + 6Ah + 6B. \tag{16}$$

Note $\lim_{h \to \mp\infty} f(h) \to \pm\infty.$

- Let α, β and γ be three zeroes of $f(h)$, then

$$\left(\frac{dh}{d\xi}\right)^2 = \frac{1}{3K}(\alpha - h)(h - \beta)(h - \gamma). \tag{17}$$

- When a zero, say δ, is real. $h = \delta$ is a constant solution. Not interested in it.

Graph of Function $f(u)$

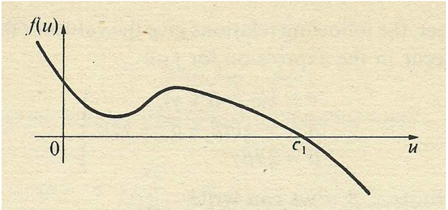


Figure: 7a Vertical axis $f(h)$. $f(h)$ with only one real zero.

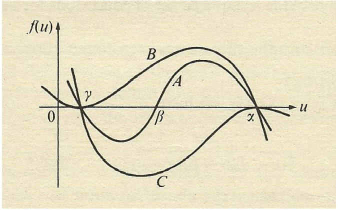


Figure: 7b Vertical axis $f(h)$. $f(h)$ with three distinct real zeroes.

Please work out your self the following results.

- We can have a real solution of the ODE only for those cases where the right hand side is positive. But the graph of $h(u)$ where $h \to \infty$ is not important.
- When there is only one real zero. Nonzero solution h of (17) is unbounded as $\xi \to \pm\infty$ - not relevant - wave can not have unbounded amplitude.
- When all three zeroes are equal, Nonzero solution $h \to \pm\infty$ at a finite value of ξ. Not relevant to us.
- When $\gamma < \beta = \alpha$, no part of the graph is relevant.
- Only case important for us is $\gamma = \beta < \alpha$, which we shall discuss.

To be rewritten

- Model equation 1:

$$\frac{dh}{d\xi} = d(\alpha - h), \quad d = \ const \ > 0.$$

In this case rhs is positive, so ξ increases and $h \to \alpha-$ at a finite value of ξ.

- Model equation 2:

$$\frac{dh}{d\xi} = -d(h - \beta), \quad d = \ const \ > 0.$$

In this case ξ decreases and $h \to \beta+$ at a finite value of ξ.

- Model equation 3:

$$\left(\frac{dh}{d\xi}\right)^2 = d(h - s)^2, \quad d = \ const \ > 0.$$

In this case $h \to s$ at as $\xi \to \mp\infty$

Travelling Waves of Permanent Form of KdV E ··· cont. Model Examples

To be rewritten

- Model equation 1:

$$\frac{dh}{d\xi} = d(\alpha - h), \quad d = \; const \; > 0.$$

Here $\alpha - h = cons.e^{d\xi}$. In this case rhs is positive, so ξ increases and $h \to \alpha^-$ at a finite value of ξ.

- Model equation 2:

$$\frac{dh}{d\xi} = -d(h - \alpha), \quad d = \; const \; > 0.$$

In this case ξ decreases and $h \to \alpha^+$ at a finite value of ξ.

- Model equation 3:

$$\left(\frac{dh}{d\xi}\right)^2 = d(h - \beta)^{1/2}, \quad d = \; const \; > 0.$$

In this case $h \to \beta$ at as $\xi \to \mp\infty$

- **Case 2:** All three zeroes are real, let $\gamma < \beta < \alpha$. $f(h) > 0$ for $\beta < h < \alpha$.

- Graph shows that value of solution $h(\xi)$ oscillates between β and α.

- When h reaches $\alpha-$ for a value of ξ, the value of h reverses its direction on the graph of h against ξ and reaches β^+ for another finite value of ξ.

- Both these values are attained at the end of a closed bounded interval, which will turn out to be period oscillation of $h(\xi)$ on ξ-axis.

- **Case 2 continued** - all three zeroes are are real.
- Define $s^2 = (\alpha - \beta)/(\alpha - \gamma)$.
- Solution is a periodic function defined on $(-\infty < \xi < \infty)$ by

$$u(x,t) \equiv h(\xi) = \beta + (\alpha - \beta)Cn^2 \left[\xi\sqrt{\frac{\alpha - \gamma}{12K}}, s\right] \tag{18}$$

where Cn is Jacobian elliptic function, which is a periodic function. ξ-period of $u(x,t)$ is

$$P = 4\sqrt{\left(\frac{3K}{\alpha - \beta}\right)}K(s^2), \tag{19}$$

where $K(s^2)$ is complete elliptic integral.

Here we get a bounded periodic solution, period defined above. K&dV called it **Cnoidal wave.**

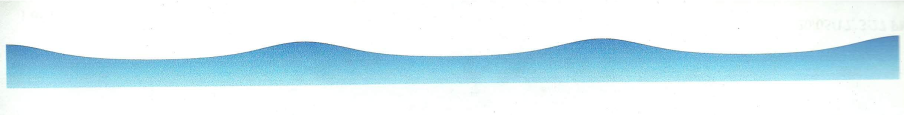


Figure: Cnoidal wave on the surface of water.

Case 2: Cnoidal Wave Experimentally Observed

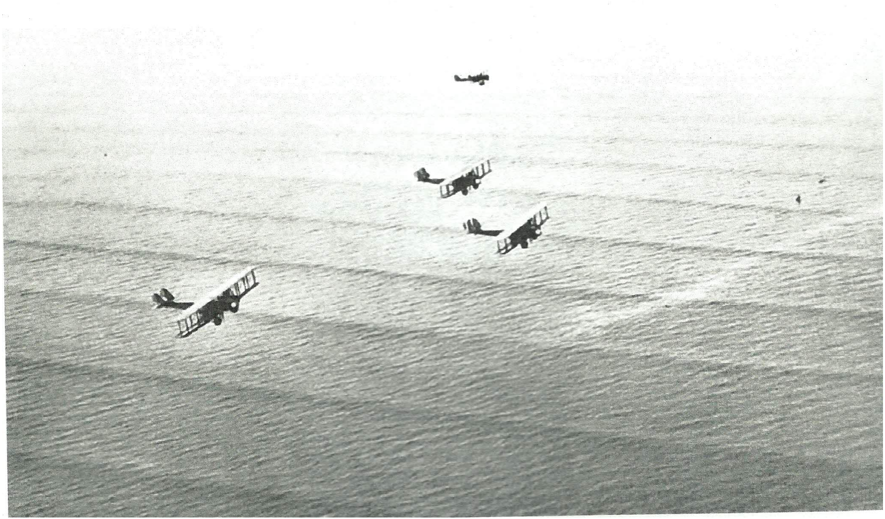


Figure: Cnoidal wave observed by US airforce.

Case 2: Cnoidal Wave Features Over one Period

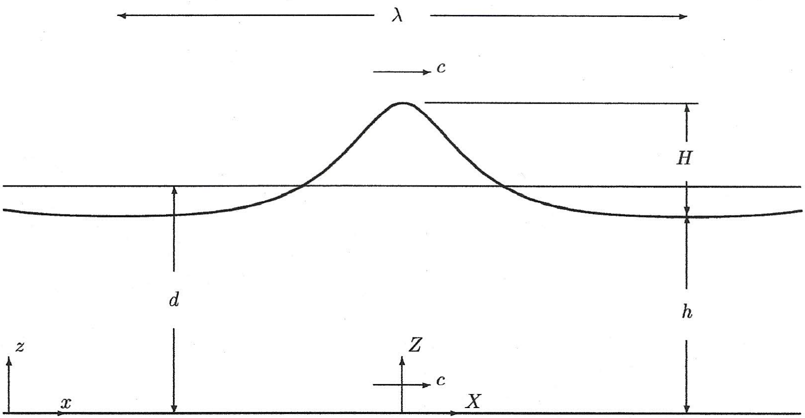


Figure: 10. Graph of $Cn(x)$ looks almost like a $\cos(x)$ with maximum and minimum values 1 and -1. Cnoidal wave Features over one period, h is depth of undisturbed water, λ is the wavelength and H is the hight of the wave.

A limiting case of Cnoidal wave - Case 2A:

- When β is very close to α but $\alpha - \gamma$ is not small, set $\frac{\alpha-\beta}{\alpha-\gamma} \equiv s^2 = \varepsilon$.
- For small ε,

$$Cn^2\left[\xi\sqrt{\frac{\alpha-\gamma}{12K}}, s\right] \approx \cos^2\left[\xi\sqrt{\frac{\alpha-\gamma}{12K}}, s\right] \quad (20)$$

and the cnoidal wave solution (18) becomes

$$u(x,t) \approx \alpha - (\alpha-\beta)\sin^2\left[\xi\sqrt{\frac{\alpha-\gamma}{12K}}\right]. \quad (21)$$

In the limit $\beta \to \alpha-$ period tends to

$$P \to 2\pi\sqrt{\frac{3K}{\alpha-\gamma}}. \quad (22)$$

- The only case left out for $f(\xi)$ to be positive is **Case C:** $\gamma = \beta < \alpha$. We can choose $\gamma = u_\infty$,
- In this case, the velocity of propagation (or translation) c, of wave turns out to be

$$c = u_\infty + \frac{a}{3} \tag{23}$$

where $a = \alpha - \gamma$, is the amplitude of the wave.

- (28) becomes

$$\left(\frac{dh}{d\xi}\right)^2 = \frac{1}{3k}(\alpha - h)(h - \gamma)^2. \tag{24}$$

Solution of this is discussed in [1].

Solitary Wave Solution Expression

$$u(x,t) \equiv h(x-ct) = u_\infty + a\ sech^2\left[\sqrt{\left(\frac{a}{12K}\right)}\left\{x - \left(u_\infty + \frac{a}{3}\right)t\right\}\right] \tag{25}$$

Solitary Wave Shape

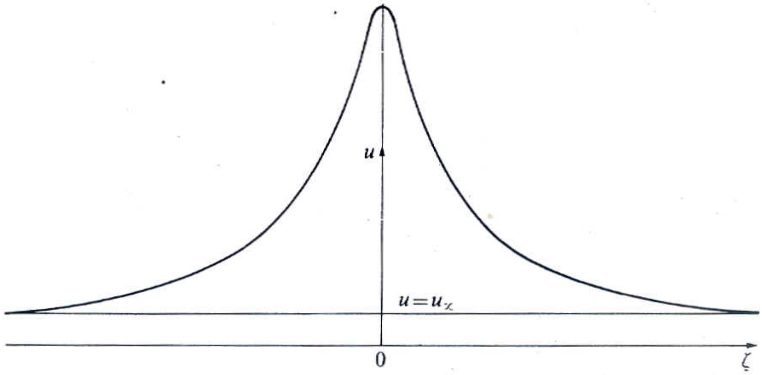


FIG. 2.7. Profile of a solitary wave joining constant states at $\xi = \pm\infty$ and localized in ξ.

Figure: Solitary Wave - an appropriate name given by Scott-Russell. In this lecture, by solitary waves we shall refer only to KdV solitary wave (SW).

Solitary Wave Solution Characteristics

- Velocity of SW relative to uniform velocity u_∞ at infinity is proportional to amplitude.
- Its width $2\pi\sqrt{\frac{12K}{a}}$ is inversely proportional to square root of amplitude.
- Its width is proportional square root of dispersion coefficient K, whose role is to spread the wave profile. In this process K balances the accumulating effect of nonlinearity.
- Amplitude a is independent of the uniform velocity at infinity on both sides.

Superpostion of Linear Waves

1. Sum $u_1 + u_2$ of two solutions u_1 and u_2 of a linear homogeneous equation is also a solution.
2. This means if there are two localised waves governed by linear equations and moving in opposite directions interact and come out unchanged later.
3. Can you imagine this also for waves moving in same direction?
4. What do you for waves governed by a wave equation

$$(\frac{\partial}{\partial t} + 5\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x})u = 0? \tag{26}$$

Soliton

1. This is not imaginable for waves governed by nonlinear equations as principle of linear superposition is not valid.
2. KdV solitary wave would not have attracted so much attention but for a special property first observed in numerical experiments in 1965.
3. Note that a bigger solitary wave moves faster than a smaller solitary wave
4. It was observed that
   "a bigger solitary wave initially behind a smaller one, overtakes, interacts and moves ahead. Both emerge unchanged, smaller one moving behind the bigger one".

Interaction of Two Solitons

The method of solution of KdV equation and the properties of solutions, are amongst most important developments in mathematics in 20th century. First discovered numerically by Zabusky and Kruskal in 1965.

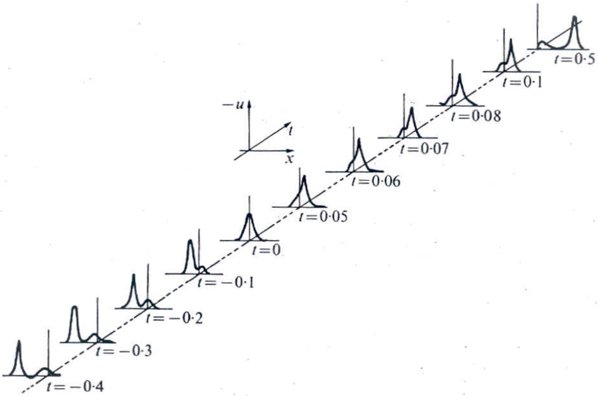


FIG. 3.1. Interaction of two solitons.

Persitence and reimergence led to coining of a new word **soliton**, which was **later** found to appear in solutions of many equations.

References Page 1

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Thank You for Your Attention!