

KdV Equation - Lecture 2 & 3

Inverse Scattering Method

Phoolan Prasad



DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE, BANGALORE

Science Academies Refresher Course
Partial Differential Equations and their Applications
July 03 - July 15, 2017

August 1, 2019

Solitary Wave Solution of KdV E

When we consider KdV E in the form

$$u_t + uu_x + Ku_{xxx} = 0, \quad K > 0, \quad (1)$$

the solitary wave solution, with $c = u_\infty + \frac{a}{3}$, is

$$u(x, t) = u_\infty + a \operatorname{sech}^2 \left[\sqrt{\left(\frac{a}{12K}\right)} \left\{ x - \left(u_\infty + \frac{a}{3}\right) t \right\} \right] \quad (2)$$

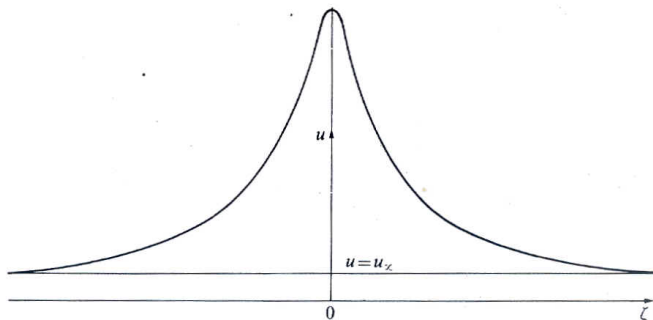


FIG. 2.7. Profile of a solitary wave joining constant states at $\xi = \pm \infty$ and localized in ξ .



Solitary Wave Solution of KdV E Which Decays to Zero at Infinity

When we take solitary wave solution of KdV E which $\rightarrow 0$ as $x \rightarrow \mp\infty$, we get

$$u(x, t) = a \operatorname{sech}^2 \left[\sqrt{\left(\frac{a}{12K}\right)} \left\{ x - \frac{a}{3}t \right\} \right] \quad (3)$$



Transformation of KdV E Used in Inverse Scattering Method

We make following transformation:

$$x = K^{1/3}x', \quad u = -6K^{1/3}u', \quad t = t'$$

then (1) becomes

$$u'_{t'} - 6u'u'_{x'} + u'_{x'x'x'} = 0. \quad (4)$$

Solution (3) becomes

$$u'(x, t) = -\frac{A^2}{2} \operatorname{sech}^2 \left[\frac{A}{2}(x' - A^2t') \right], \quad (5)$$

where we have used $A^2 = \frac{a}{3K^{1/3}}$.



Convenient Form of KdV E Used for Further Development of Theory and Method of Solution

We remove ' from the new variables. We shall now use KdV E in the form

$$u_t - 6uu_x + u_{xxx} = 0. \quad (6)$$

Solitary wave solution vanishing at infinity is

$$u(x, t) = -\frac{A^2}{2} \operatorname{sech}^2 \left[\frac{A}{2}(x - A^2 t) \right], \quad (7)$$

where $-\frac{A^2}{2}$ is amplitude and A^2 is velocity of propagation of the solitary wave.

Note: This gives an inverted shape of a solitary wave but still moving in positive x -direction.



Integral of Motion of Motion

An **integral of motion** of an equation of evolution is a function $I(u)$, which remains constant as solution u evolves.

KdV E has infinity of integrals of motion. We shall meet many of these in this lecture.



Conservation laws and Integrals of Evolution

KdV E has infinity of conservation laws of form

$$T_t + X_x = 0 \quad (8)$$

where T , conserved quantity, and $-X$, flux, are functions of u and derivatives of u wrt to x only and not t .

- Conservation laws are important to derive **integrals of motion** and many other properties.
- If flux $X \rightarrow 0$ as $|x| \rightarrow \infty$ sufficiently rapidly, then

$$\frac{d}{dt} \int_{-\infty}^{\infty} T dx = X|_{\infty} - X|_{-\infty} = 0$$

which implies $\int_{-\infty}^{\infty} T dx = \text{const.}$

- Thus $\int_{-\infty}^{\infty} T dx$ is an integral of motion.

We shall not pursue it further but go to method of solution.



For periodic solution of KdV E, limits can be two ends of a period i.e.,

$$\int_{-\lambda/2}^{\lambda/2} T dx = \text{constant}$$

where λ is the period.

First three of infinity of conservation laws from KdV E:

- 1 $u_t + (-3u^2 + u_{xx})_x = 0,$
- 2 $(u^2)_t + (-4u^3 + 2uu_{xx} - u_x^2)_x = 0,$
- 3 $(u^3 + \frac{1}{2}u_x^2)_t + (-\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \frac{1}{2}u_{xx}^2)_x = 0.$



Physical interpretation of integrals of evolution obtained from conservation laws:

- For water waves, first integral of evolution $\int_{-\infty}^{\infty} u dx$ represents conservation of mass of water above the constant depth.
- Second integral of evolution $\int_{-\infty}^{\infty} u^2 dx$ represents conservation of horizontal momentum.
- Third integral of evolution $\int_{-\infty}^{\infty} (u^3 + \frac{1}{2}u_x^2) dx$ represents conservation of energy.

We shall not pursue it further but go to method of solution.



Schroedinger Equation Associated with KdV E

Solution of initial value problem KdV E had eluded mathematicians for more than 60 years since it was derived.

- But for a class of solutions decaying rapidly at infinity, a very innovative method was discovered by Gardner, Greene, Kruskal and Miura (1967)(see for review [6]).
- In this method we consider the time-independent Schroedinger equation (SE) which is **linear** containing solution u of KdV E:

$$\psi_{xx} + \{\lambda - u(x, t)\} \psi = 0, \quad \lambda = \textit{independent of } x \quad (9)$$

and t appears only as a parameter.

We may suppress dependence of u on t and consider eigenvalue problem for (9).



Eigenvalue Problem for SE

- For u tending to 0 sufficiently rapidly at $\pm\infty$, solve

$$\psi_{xx} + \{\lambda - u\} \psi = 0 \quad (10)$$

for smooth ψ with conditions

1

$$\psi \rightarrow 0 \text{ as } x \rightarrow \mp\infty \quad (11)$$

2 or

$$\psi \rightarrow a(k) \exp(-ikx) \text{ as } x \rightarrow -\infty, \quad (12)$$

$$\psi \rightarrow \exp(-ikx) + b(k) \exp(ikx) \text{ as } x \rightarrow \infty \quad (13)$$

where k is constant with respect to x .

- Values of λ for which conditions (11) or (12) and (13) are satisfied are defined as **eigenvalues** of boundary value problems in (10) - (13).
- All eigenvalues constitute **spectrum** of eigenvalue problem.



Discrete and Continuous Spectrum

We state some results without proof, well studied in literature:

- Spectrum of eigenvalue problem (10) and (11) contains discrete eigenvalues which are negative. We denote them as

$$\begin{aligned} \lambda &= -\kappa_1^2, -\kappa_2^2, \dots, -\kappa_n^2; & \kappa_i > 0; \\ -\kappa_1^2 &< -\kappa_2^2 < \dots < -\kappa_n^2; \end{aligned} \quad (14)$$

- Eigenvector (to be shown unique) corresponding to $-\kappa_i^2$ has following behaviour at infinity

$$\psi_i(x) \rightarrow c'_i e^{\kappa_i x} \text{ as } x \rightarrow -\infty; \quad \psi_i(x) \rightarrow c_i e^{-\kappa_i x} \text{ as } x \rightarrow \infty. \quad (15)$$

- Spectrum of eigenvalue problem (10) with (12) and (13) is continuous and positive

$$\lambda = k^2, \quad k > 0. \quad (16)$$

Physical Interpretation from Quantum Mechanics

In quantum mechanics we see following interpretation of functions appearing in (9) when u decays to zero sufficiently rapidly:

- ψ is wave function of a moving particle under an external field whose potential energy is u .
- λ is energy eigenvalue.
- In classical mechanics, a particle with energy E will not be able to penetrate a region where $E < u(x)$.
- In quantum mechanics a particle may be found in region where $E < u(x)$, though the probability, denoted by $|\psi|$, of finding the particle is small (but nonzero) and it rapidly tends to zero as the distance into such regions increases.

What interpretation for continuous spectrum?



Physical Interpretation 2

For continuous spectrum, wave function ψ is “spatially dependent part $\exp(-ikx)$ ” of a steady stream of plane wave $\bar{\psi}$ being sent from ∞ to interact with potential u .

$$\bar{\psi} = e^{-ikx-i\omega t} = \psi e^{-i\omega t}$$

- Result of interaction consists of reflection $b(k) \exp(ikx)$ of a part going to ∞ and remainder $a(k) \exp(-ikx)$ transmitted through potential, represented by (11) and (12) i.e.,

$$\psi \rightarrow a(k) \exp(-ikx) \text{ as } x \rightarrow -\infty, \quad \text{and} \quad (17)$$

$$\psi \rightarrow \exp(-ikx) + b(k) \exp(ikx) \text{ as } x \rightarrow \infty, \quad (18)$$

where e^{-ikx} and e^{ikx} represent left-going and right-going waves respectively. **How?**



- **Law of conservation of energy:**

energy of incident wave = energy transmitted + reflected waves

gives
$$1 = |a|^2 + |b|^2. \quad (19)$$

We shall use (19) later in mathematical formulation. It also works as normalization of ψ .



Inverse Scattering Method for KdV E

Now, as in many cases such as “shock”, we borrow the word “potential” from physics for u in mathematics.

- This method, appropriately called **Inverse scattering transform method - ISTM**, is method for solving some **non-linear** PDEs.
- It is one of most important developments in mathematical physics in last century. Method is a non-linear analogue, and in some sense generalization, of the Fourier transform (FT), which is applied to solve many **linear PDEs**.
- In order to convince beginners of this analogue and help him to understand ISTM, we first sketch FTM.



Fourier Transform Method for Linear PDEs

- Consider an initial value problem for a linear evolution equation:

$$u_t = L(D)u, \quad D = \frac{\partial}{\partial x}, \quad (20)$$

$$u(x, 0) = u_0(x). \quad (21)$$

Substituting of $u = e^{i(kx - \omega t)}$ in (20), we get

$$\omega = iL(ik) \quad (22)$$

- Assume that the linear operator L is such that

$$\Omega(k) := iL(ik) \text{ is a real function of } k. \quad (23)$$

Then equation (19) leads to a real dispersion relation

$$\omega = \Omega(k). \quad (24)$$

Problem: Compare dispersion relations when $L(D) = D^2$ and $L(D) = D^3$.



Conditions on u not mentioned.

- FT $\hat{u}(k, t)$ of $u(x, t)$ and inverse FT of $\hat{u}(k, t)$ leading to recovery of $u(x, t)$, are defined respectively as

$$\hat{u}(k, t) := \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx, \quad u(x, t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k, t) e^{ik\xi} d\xi. \quad (25)$$

Note that in these two equations, t only plays role of a parameter.

- To solve IVP (20 -21), take FT of these equations note that $FT(Du) = ik\hat{u}$, to get

$$\frac{d\hat{u}}{dt} = -i\Omega(k)\hat{u}, \quad (26)$$

$$\hat{u}(k, 0) = FT(u_0(x)) = \hat{u}_0(k), \quad \text{say.} \quad (27)$$



Fourier Transform (FT) Method ... conti.

- Crucial steps involved in FT method are
 - ① to solve initial value problem (26-27) and
 - ② to recover solution $u(x, t)$ using the inverse FT formula i.e., second part of (25).
- Solution of (26-27) is

$$\hat{u}(k, t) = \hat{u}_0 e^{-i\Omega(k)t}. \quad (28)$$

FT method is schematically shown in Figure 12

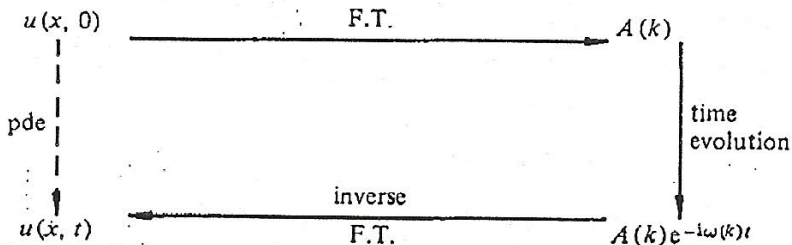


Figure: 12 FT Method, here $A(k) = \hat{u}_0(k)$ and $\omega(k) = \Omega(k)$.



Discrete Eigenvalues of Schoedinger Equation (SE)

We arrange the finite number of discrete eigenvalues as

$$\lambda = -\kappa_1^2 < -\kappa_2^2, < \dots, < -\kappa_n^2; \quad \kappa_i > 0. \quad (29)$$

We state a

Theorem

A discrete eigenvalue λ is simple i.e., the eigenvector ψ corresponding to it is unique except for a constant multiplying factor.

Proof.

Let ψ_1 and ψ_2 be two eigenvectors, then

$$\begin{aligned} \frac{\psi_{1xx}}{\psi_1} = u - \lambda = \frac{\psi_{2xx}}{\psi_2} &\implies (\psi_{1x}\psi_2 - \psi_{2x}\psi_1)_x = 0 \\ \psi_{1x}\psi_2 - \psi_{2x}\psi_1 = \text{const.} &= 0, \text{ since } \psi_{1,2} \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{aligned}$$

From this it follows we can derive $\psi_1 = \text{constant times } \psi_2$. □

Continuous Eigenvalues Are Degenerate

- For a continuous eigenvalue k^2 in the spectrum, we have already seen two independent solutions $\rightarrow \exp(\pm ikx)$ at infinity.



Normalysing Eigenvectors of SE

Discrete eigenvalue $-\kappa^2$:

- We have mentioned that m th eigenvector ψ_m of discrete eigenvalu $-\kappa_m^2$ vanishes at infinity sufficiently rapidly, which we shall see when we calculate ψ .
- We make eigenvector ψ_m unique by normalysing it by

$$\int_{-\infty}^{\infty} \psi_m^2 dx = 1, \quad (30)$$

which determines c_m and c'_m .

Continuous eigenvalue k^2 :

- We used from physics conservation of energy to derive

$$|a|^2 + |b|^2 = 1. \quad (31)$$

- But this can be deduced purely by mathematical steps, see deivation of (3.9), page 43 in [2].



Scattering Data of u

- Set $S(t) := \{\kappa_1, \dots, \kappa_n; c_1, \dots, c_n; a(k), b(k)\}$ is called **scattering data** associated with potential u .
- The scattering data for u is found by solving time independent eigenvalue problem for SE.
- In eigenvalue problem, t appearing in $u(x, t)$ is only a parameter. All elements of scattering data depend on t .



We look at one natural problem and an unusual question:

- 1 Direct Scattering Problem - Given u , find the scattering data by solving the eigenvalue problem for SE.
- 2 Question - Given scattering parameters of u , can we find u ?

Problem in (2) was solved (before 1950 [2], author?) and above question was answered positively - known as **inverse Scattering Method** by Gel'fand and Levitan (1951) [4] and Marcenko (1957).



Scattering Transform Method

- Now we ask another question. When $u(x, t)$ evolves according to KdV E, can we determine evolution of scattering data?
- If the answer is positive, we can formulate now scattering transform method [see figure](#)

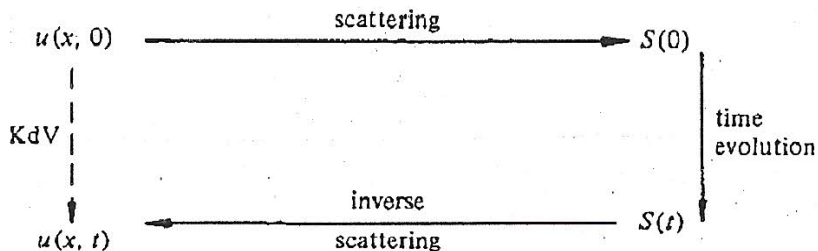


Figure: 13 ST method is similar to FT method. $S(0)$ and $S(t)$ are scattering data at $t = 0$ and t .



- To complete ST method, we just need to find a method to find evolution of $S(0)$ to $S(t)$.
- This was answered positively in 1967 by Gardner et al [3].
- We describe the complete method once more and fill in the gaps in steps.



KdV - Nature of Initial Data

We consider IVP

$$u_t - 6uu_x + u_{xxx} = 0 \quad (32)$$

$$u(x, 0) = u_0(x). \quad (33)$$

Assume that initial data u_0 satisfies conditions:

$$\sum_{i=0}^4 \int_{-\infty}^{\infty} \left| \frac{\partial^i u_0(x)}{\partial x^i} \right|^2 dx < \infty. \quad (34)$$

Bona¹ and Smith (1975) showed that it ensures existence of a classical solution of IVP.

$$\int_{-\infty}^{\infty} (1 + |x|) |u_0(x)| dx < \infty. \quad (35)$$

¹A fried of mine



Faddeev (1958) showed that (35) ensures existence of a solution of eigenvalue problem of SE. This rules out **data already like cnoidal wave - not vanishing up to $\pm\infty$** .

Lax (1968) proved uniqueness of the above IVP.

Lax's paper, containing many more mathematical ideas, opened up new areas of research both in pure and applied mathematics.

Inverse scattering method of KdV E during 1967-68 and general theory of Lax (1968) together form one of the most important developments in mathematics in 20th century.



- As in case with FT method, associate with u a suitable transform.
- In this case the transform of u is not just one function but a set of functions.
- This set $S(t)$ consists of eigenvalues and eigenfunctions of time independent Schroedinger operator

$$\frac{d^2\psi}{dx^2} + (\lambda - u)\psi$$

- For an arbitrary u satisfying the conditions (34-36), it is not easy to find the evolution of $S(t)$ from $S(0)$.
- However, since $u(., t) \rightarrow 0$ sufficiently rapidly at infinity, it is easy to determine $S(t)$, see [1] and [2].



Reminder - Normalization of Eigenvectors

- For discrete eigenvalue $\lambda = -\kappa_i^2$, we normalized the eigenvector by ψ_i in (30) by

$$\int_{-\infty}^{\infty} \psi_i^2 dx = 1$$

- For continuous eigenvalue k^2 we normalized the eigenvector in (30) by

$$|a|^2 + |b|^2 = 1.$$



Solution $u(x, t)$ of KdV E evolves with time but so far in eigenvalue problem for SE, we did not show dependence of $S(t)$ on t .

For a discrete eigenvalue $\lambda(t) = -\kappa_i^2(t)$, normalised eigenvector

$$\psi_i(x, t) \rightarrow c_i(t)e^{-\kappa_i(t)x} \text{ as } x \rightarrow \infty.$$

For a continuous eigenvalue $\lambda(t) = k^2(t)$ normalised eigenvector satisfies

$$\psi(x, t) \rightarrow a(k, t) \exp(-ikx) \text{ as } x \rightarrow -\infty,$$

and

$$\psi(x, t) \rightarrow \exp(-ikx) + b(k, t) \exp(ikx) \text{ as } x \rightarrow \infty.$$

Now we state Evolution Theorems without proof, for proof see [1] and [2].



Theorem

Theorem 1: When $u(x, t)$ evolves according to KdV E, the discrete eigenvalues are independent of t and a continuous eigenvalue may be assumed to be constant. this means

$$\kappa_i(t) = \kappa_i(0) \quad \text{and} \quad k(t) = k(0). \quad (36)$$

Theorem

Theorem 2: When $u(x, t)$ evolves according to KdV E, evolution of $c_i(t)$, $a(k, t)$ and $b(k, t)$ is given by

$$c_i(t) = c_i(0)e^{4k_i^3 t}, \quad (37)$$

$$a(k, t) = a(k, 0) \quad \text{and} \quad (38)$$

$$b(k, t) = b(k, 0)e^{8ik^3 t}. \quad (39)$$

Review So Far

- 1 KdV E has solitary wave solution, which was named “Soliton” for its special interaction property.
- 2 Challenge: a) To find solution having n solitons and see the interaction property. b) To solve IVP.
- 3 Direct solution not possible. To develop a transform method similar to FTM.
- 4 It has been possible by associating Schrödinger equation

$$\psi_{xx} + (\lambda - u(x, t))\psi = 0$$

where $u(x, t)$ is a solution of the KdV E. How time independent?

- 5 To find scattering data
 $S(t) := \{\kappa_1, \dots, \kappa_n; c_1(k_1, t), \dots, c_n(k_n, t); a(k, t), b(k, t)\}$ at $t=0$.
- 6 Evolution of scattering data completed and got $S(t)$.
- 7 To construct $u(x, t)$ from $S(t)$.



Solution of KdV E IVP Main Steps

- Given initial data $u_0(x)$, by direct scattering method, we can find initial scattering data $S(0)$.
- From theorems on last slides, we can determine scattering data $S(t)$.
- Inverse scattering method was already solved about 16 years before it was required by Gardaner et al. for KdV E.
- We only describe 3 main steps without any proof.



Solution of KdV E IVP Main Steps

Scattering data consists of a number of functions and we have to recover just one function of x and t .

Step 1: Given

$S(t) := \{\kappa_1, \dots, \kappa_n; c_1(k_1, t), \dots, c_n(k_n, t); a(k, t), b(k, t)\}$, we define function $F(\xi; t)$ of transform variable ξ :

$$F(\xi; t) = \sum_{i=1}^n c_i^2(\kappa_i, t) e^{-\kappa_i \xi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k, t) e^{ik\xi} dk \quad (40)$$

$$= \sum_{i=1}^n c_i(0) e^{(8\kappa_i^3 t - \kappa_i \xi)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k, 0) e^{i(8k^3 t + k\xi)} dk. \quad (41)$$

This function plays the role of Fourier transform function $\hat{u}(\xi, t)$. Notice that all $\kappa_1, \dots, \kappa_n; k \in \mathbb{R}_+$ join together to give one transform variable ξ and they disappear.



We first transform back to a function $K(x, z; t)$ of two independent variables, t is simply a parameter here.

Step 2: In this step we solve Gel'fand-Levitan (and also Marcenko) equation for $K(x, z)$ (in integral equation x and t appear only as parameters)

$$K(x, z; t) + F(x + z; t) + \int_x^\infty K(x, y; t)F(y + z; t)dy = 0. \quad (42)$$

Note that in the solution ξ disappears in integration process with respect to y .



Step 3: In this step we find solution of IVP for KdV E and complete inverse scattering transform method

$$u(x, t) = \left\{ -2 \frac{d}{dx} K(x, z; t) \right\} \Big|_{z=x} = -2 \frac{d}{dx} K(x, x; t). \quad (43)$$

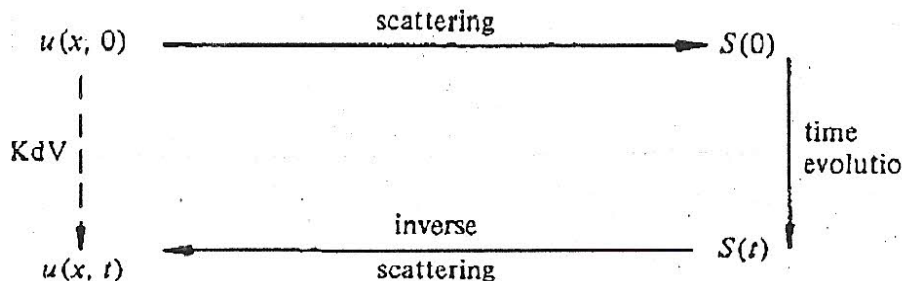
- Steps 2 and 3 are most uninteresting part of this lecture for mathematicians - they need to learn these steps.
- Proof will require too much of time.
- My aim here is just to show a very interesting and important area of study in a course in PDE.

ISTM is really a very innovative idea - our hats off to discoverers of the method - Gardner, C. S., Greene, J.M., Kruskal, M. D. and Miura, R. M.



Schematic diagram Scattering Transform Method

We have completed description (not proof) of Scattering Transform Method, which I again show in the following diagram.



13 ST method is similar to FT method. $S(0)$ and $S(t)$ are scattering data at $t = 0$ and t .



Example 1: One Soliton Solution of KdV E

We produce the steps without details of calculations.

Assume the initial data as a soliton:

$$u_0(x) = -2 \operatorname{sech}^2 x. \quad (44)$$

Initial scattering data $S(0)$ is obtained by solving the eigenvalue problem for

$$\psi_{xx} + (\lambda + 2 \operatorname{sech}^2 x)\psi = 0 \quad (45)$$

It can be shown that the problem has only one discrete eigenvalue $\lambda = -\kappa_1^2 = -1$ with eigenfunction

$$\psi = \frac{1}{\sqrt{2}} \operatorname{sech} x = \frac{\sqrt{2}}{e^{-x} + e^x} \rightarrow \sqrt{2}e^{-x} \text{ as } x \rightarrow \infty. \quad (46)$$



Example 1: One Soliton Solution of KdV E ... conti.

This implies

$$c_1(0) = \sqrt{2}, \quad c_1(t) = \sqrt{2}e^{4t} \quad (47)$$

and it can be further shown that this u_0 is a **reflectionless potential** (a potential u for which $b(k) = 0$), which means

$$b(k) = 0, \quad a(k) = 1. \quad (48)$$

$$\text{Thus } F(\xi; t) = 2e^{8t-\xi}. \quad (49)$$

Gel'fand-Levitan, Marchenko equation becomes

$$K(x, z; t) + 2e^{8t-(x+z)} + 2 \int_x^\infty K(x, y; t)e^{8t-(y+z)} dy = 0. \quad (50)$$

In this integral equation x and t appear only as parameters.



Example 1: One Soliton Solution of KdV E ... conti.

Multiplying by e^z

$$K(x, z; t)e^z + 2e^{8t-x} + 2 \int_x^\infty K(x, y; t)e^{8t-y} dy = 0. \quad (51)$$

Set $K(x, z; t)e^z = L$, then

$$L + 2e^{8t-x} + 2e^{8t} \int_x^\infty Le^{-2y} dy = 0. \quad (52)$$

Since 2nd and 3rd terms are independent of z , so must be L , hence $K(x, z; t)e^z = L(x, t)$. Hence y does not appear in integrand Le^{8t} and we get

$$L(x, t) = -\frac{2e^{8t-x}}{1 + e^{8t-2x}} \quad \text{so that} \quad K(x, z; t) = -\frac{2e^{8t-x-z}}{1 + e^{8t-2x}}. \quad (53)$$



Example 1: One Soliton Solution of KdV E -Final Step ... conti.

Solution of IVP is

$$u(x, t) = -2 \frac{\partial}{\partial x} K(x, x; t)|_{t=const} = 2 \frac{\partial}{\partial x} \left(\frac{2e^{8t-2x}}{1 + e^{8t-2x}} \right)$$

Hence finally

$$u(x, t) = -2 \operatorname{sech}^2(x - 4t) \quad (54)$$

This is solitary wave of amplitude **2** (amplitude in the standing solitary wave prescribed in the initial data) with velocity of propagation **4**.

Here $\kappa_1^2 = 1$, hence we do an extrapolation, **2** and **4t** are replaced by **$2\kappa_1^2$** and **$4\kappa_1^2 t$** respectively.

$$u(x, t) = -2 \operatorname{sech}^2(x - 4t) = -2\kappa_1^2 \operatorname{sech}^2(x - 4\kappa_1^2 t). \quad (55)$$



Example 1: One Soliton Solution of KdV E - an Observation

Comparing $\psi_{xx} + (\lambda + 2 \operatorname{sech}^2 x)\psi = 0$ and

$$\psi_{xx}(x; t) + (\lambda + 2 \operatorname{sech}^2(x - 4t))\psi = 0,$$

$$\psi(x; t) = \frac{1}{\sqrt{2}} \operatorname{sech}(x - 4\kappa_1^2 t), \quad \kappa_1(t) = \kappa_1(0) = 1. \quad (56)$$

We see for one soliton solution

$$u(x, t) = -4\kappa_1 \psi_1^2(x, t).$$

We **verify** now an **astounding important** possibility for n soliton solution, (though such an expression has not been obtained),

$$u(x, t) = -4 \sum_1^n \kappa_i \psi_i^2(x, t). \quad (57)$$



Example 2: Two Soliton Solution of KdV E

In this case we produce still less steps without details of calculations. For details see [1]

Assume the initial data as a soliton:

$$u_0(x) = -6 \operatorname{sech}^2 x, \quad (58)$$

in which amplitude and width of pulse do not match with the solitary wave solution.

Initial scattering data $S(0)$ is obtained by solving the eigenvalue problem for

$$\psi_{xx} + (\lambda + 6 \operatorname{sech}^2 x)\psi = 0 \quad (59)$$

Potential $u_0(x)$ is again reflectionless with only two discrete eigenvalues $\lambda = -4$ (which gives $\kappa_1 = 2$) and $\lambda = -1$ (which gives $\kappa_2 = 1$).



A bit long procedure gives solution of the IVP of the KdV E as

$$u(x, t) = -12 \frac{3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)}{[\cosh(3x - 36t) + 3 \cosh(x - 28t)]^2}. \quad (60)$$

This expression does not reveal much about the wave - it has too many expressions with arguments consisting of four forms $\alpha t - \beta x$.

But it contains a most fascinating result discovered in 1967.

Discovery of first soliton in (60)

With $\xi = x - 16t = x - 4\kappa_1^2 t$, we write above expression in form

$$u(x, t) = -12 \frac{3 + 4 \cosh(2\xi + 24t) + \cosh(4\xi)}{[\cosh(3\xi + 12t) + 3 \cosh(\xi - 12t)]^2}. \quad (61)$$



Example 2: First Soliton in Solution

We take limit of (61) as $t \rightarrow \infty$.

When $t \gg 1$

$$\cosh(2\xi + 24t) = \frac{1}{2} \{ e^{(2\xi+24t)} + e^{-(2\xi+24t)} \} \approx \frac{1}{2} e^{(2\xi+24t)}. \quad (62)$$

Using similar expression for other *cosh* functions and neglecting terms of $O(1)$ in comparison with powers of e^t , we get

$$\begin{aligned} \lim_{x \rightarrow \infty} u(x, t) &= -96 \frac{e^{2\xi}}{9e^{-4\xi} + 6e^{2\xi} + e^{6\xi}} \\ &= 32 \frac{1}{\left(\frac{1}{\sqrt{3}} e^{2\xi} + \sqrt{3} e^{-2\xi} \right)^2} \\ &= 32 \frac{1}{\left(\frac{1}{\sqrt{3}} e^{2\xi - 2\xi_1} + \sqrt{3} e^{-2\xi + 2\xi_1} \right)^2}, \end{aligned} \quad (63)$$

where $e^{4\xi_1} = 3$.



Example 2: First Soliton in Solution ... conti.

Finally we get ($\kappa_1 = 2$)

$$\begin{aligned}\lim_{x \rightarrow \infty} u(x, t) &= -8 \operatorname{sech}^2\{2(2\xi - 2\xi_1)\}, \\ &= -8 \operatorname{sech}^2\{2(x - 16t - 2\xi_1)\}, \\ &= -2\kappa_1^2 \operatorname{sech}^2\{\kappa_1(x - 4\kappa_1^2 t - 2\xi_1)\},\end{aligned}\quad (64)$$

$$\text{where } \xi_1 = \frac{1}{4} \ln 3. \quad (65)$$

Similarly we can show

$$\lim_{x \rightarrow -\infty} u(x, t) = -2\kappa_1^2 \operatorname{sech}^2\{\kappa_1(x - 4\kappa_1^2 t + 2\xi_1)\}, \quad \kappa_1^2 = 4. \quad (66)$$

(64) and (66) show that a soliton, which starts at a large distance on negative side of x -axis emerges in the solution later at a large distance on positive side of x -axis **but with a phase shift.**



Example 2: Second Soliton in Solution

Consider 2nd discrete eigenvalue $\lambda = -\kappa_2^2 = -1$ Finally we get

$$\lim_{x \rightarrow -\infty} u(x, t) = -2 \operatorname{sech}^2(x - 4t - \xi_2) \quad (67)$$

$$= -2\kappa_2^2 \operatorname{sech}^2\{\kappa_2(x - 4\kappa_2^2 t - \xi_2)\}, \quad (68)$$

$$\text{where } \xi_2 = \frac{1}{2} \ln 3. \quad (69)$$

and

$$\lim_{x \rightarrow \infty} u(x, t) = -2\kappa_1^2 \operatorname{sech}^2\{\kappa_1(x - 4\kappa_1^2 t + \xi_2)\}, \quad \kappa_1^2 = 1. \quad (70)$$

(67) and (68) show that , as in the case of first soliton, second soliton starts at a large distance on negative side of x -axis emerges in the solution later at a large distance on positive side of x -axis but with a phase shift.



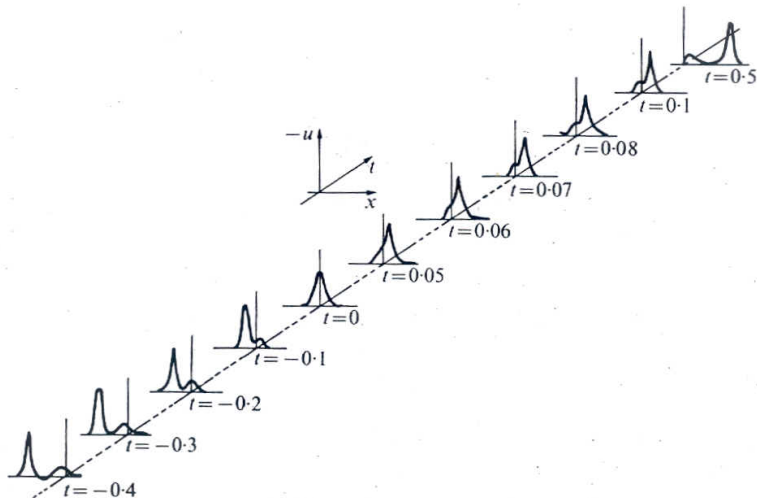
Example 2: Interaction of 2 Solitons and Their Emergence

- Above analysis shows that solution (60) contains two solitons at large distance on netive x -axis and same two solitons at large distance on positive x -axis.
- At large negative-time, the bigger soliton with amplitude 8 is behind the smaller soliton with amplitude 2.
- Bigger one moves faster than the smaller one and overtakes the smaller one.
- Interaction is given by the solution (60) at any time. Figure is drawn from graph of $u(x, t)$, shows that bigger one swallows the smaller one.
- After some time smaller one comes out but is left behind. At large positive-time, they emerge unchanged but with phase shifts $4\xi_1$ and $2\xi_2$.



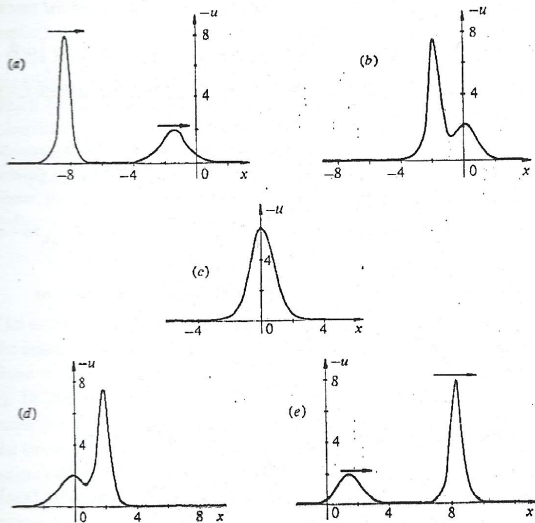
Interaction of Two Solitons Fig. 1

Interaction of two solitons is shown in (x, t, u) -space ($-u$ axis is vertically upward):



Interaction of Two Solitons Fig. 2

Fig. 4.3 The two-soliton solution with $u(x, 0) = -6 \operatorname{sech}^2 x$ (see (c)); (a) $t = -0.5$; (b) $t = -0.1$; (d) $t = 0.1$; (e) $t = 0.5$. Note that $-u$ is plotted against x .



Problem: Can we express (60) in the form

$$u(x, t) = -4\kappa_1\psi_1^2(x, t) - 4\kappa_2\psi_2^2(x, t)?$$

We have seen that such a representation is true asymptotically as $t \rightarrow \pm\infty$.

- One of great success of inverse scattering method is explicit form of n -soliton solution of KdV E by Gardner et al (1967), see [1] and [2].
- Can we express n -soliton solution in form:

$$u(x, t) = -4 \sum_1^n \kappa_i \psi_i^2(x, t)? \quad (71)$$

Here each $-4\kappa_i\psi_i^2(x, t)$ represents i th soliton.

- We show 3-soliton solution in figure on next slide.

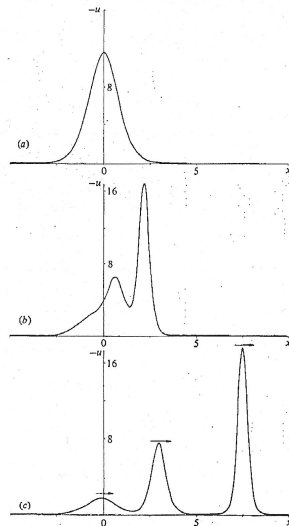


Interaction of Three Solitons

80

Initial-value problem for the KdV equation

Fig. 4.5 The three-soliton solution with $u(x, 0) = -12 \operatorname{sech}^2 x$ (see (a)); (b) $t = 0.05$; (c) $t = 0.2$. Note that $-u$ is plotted against x .



Solution When Potential is not Reflectionless

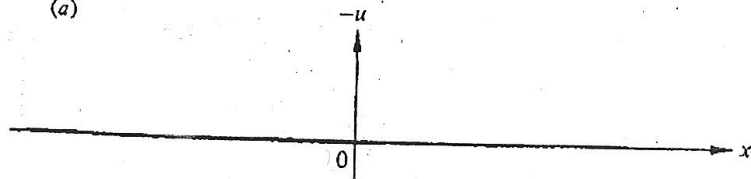
- There is no explicit solution and the solution is generally obtained by asymptotic analysis or numerical solution of the IVP.
- We present results on next few slides.



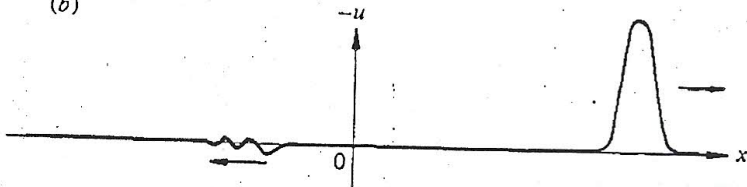
When Initial Data is a Delta Function with -ve Amplitude

Fig. 4.6 The delta-function initial profile (see (a)). The solution at a later time is sketched in (b). Note that $-u$ is plotted against x .

(a)



(b)



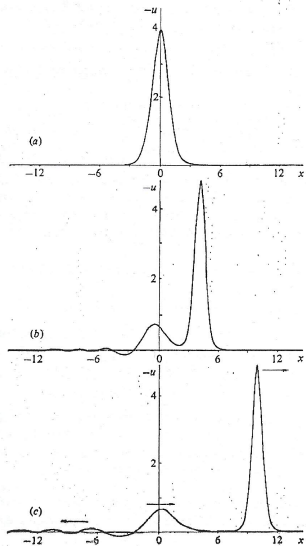
Solution When Potential is Delta Function with +ve Amplitude

- When $u_0(x) = U_0\delta(x)$, $U_0 > 0$, there is no discrete eigenvalue.
- There is no soliton. Only dispersive wave components arise for $t > 0$.
- Delta function collapses and develops dispersive wave train.
- Since this corresponds to dipression of water surface, result corresponds to Russell's obovation.
- Russell **great observation in 1844** is not only on great wave of translation but he also observed correctly interaction of two solitons, trails of oscillatory waves etc.
- **Great observational power and their interpretation - far ahead of his time.**



Two Solitons with Dispersive Wave







Fig. 4.7 Solution with two solitons and a dispersive wave, where $u(x, 0) = -4 \operatorname{sech}^2 x$ (see (a)); (b) $t = 0.4$; (c) $t = 1.0$. Note that $-u$ is plotted against x .



The i th soliton which emerges out of n -soliton solution at $\mp\infty$ corresponding to discrete eigenvalue $-\kappa_i^2$ is

$$u_i = -2\kappa_i^2 \{ \kappa_i (x - 4\kappa_i^2 t \mp \xi_i) \} \quad (72)$$



-  Bhatnagar, P. L.; *Nonlinear Hyperbolic Waves in One-dimensional Dispersive Systems*, Oxford University Press, 1979.
-  Drazin, P. G. and Johnson, R. S. *Solitons: An Introduction (Cambridge Texts in Applied Mathematics)*, 1989
-  Gardner, C. S., Greene, J.M., Kruskal, M. D. and Miura, R. M.; *Method of solving Kortewg-de Vries equation*, Phys. Rev. Lett., **19**, 1095-1097, 1967.
-  Gel'fand, I. M. and Levitan, B. M.; *On determination of a differential equation from its spectral function*, (1951), English translation - Amer. Math. Soc. Transl. (2), **1**, 253-304, 1955.
-  Korteweg, D. J. and de Vries, G., *On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves*, Philosophical Magazine, 39 (240): 422-443, 1895.
-  Robert M. Miura, *The Korteweg-de Vries Equation: A Survey of Results*, SIAM Review, **18**, 412-459, 1976.



Thank You for Your Attention!

