# A note on
# The Bicharacteristic Theorem

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**Aim of This Brief Note:** We shall first give a basic back ground of "The Bicharacteristic Theorem" and then state the theorem.

**Physical Basis of Hyperbolic Equations:** In a hyperbolic system of partial differential equations (PDEs), one independent variable plays a distinctive. We denote this **time-like** variable by $t$ and other **spatial** variables by $\boldsymbol{x} = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$. The hyperbolic nature of the system is due to the fact that the system has a sufficient number (or full set) of families of curves in the space-time which carry information with finite **speeds** from the initial plane $t = 0$ to solve an initial value problem (*a Cauchy problem*). For two independent variables ($d = 1$) these curves are **characteristic** curves and for more than two independent variables ($d > 1$) they are **bicharacteristic** curves (which can be identified with **rays** in physical $\boldsymbol{x}$-space). Single first order PDEs are the simplest examples of hyperbolic equations.
See also Chapter 2. Finite Speed of Propagation in "P. D. Lax, *On hyperbolic partial differential equations*, Courant Inst. of Math. Sci. and AMS, 2006".

**Hyperbolic System of First Order Quasilinear Equations in Multi-Space Dimensions:** We consider $n$ equations in $d+1$ independent variables $\boldsymbol{x}$ and $t$

$$A(\boldsymbol{x}, t, \boldsymbol{u}))\boldsymbol{u}_t + B^{(\alpha)}(\boldsymbol{x}, t, \boldsymbol{u})\boldsymbol{u}_{x_\alpha} + \boldsymbol{C}(\boldsymbol{x}, t, \boldsymbol{u}) = 0 \tag{1}$$

where $\boldsymbol{u} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B^{(\alpha)} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C} \in \mathbb{R}^n$.

Summation convention: A repeated symbol in subscripts and superscripts in a term will mean summation over the range of the symbol. The range of the symbols $\alpha, \beta,$ *and* $\gamma$ will be $1, 2, \cdots, d$ and those of $i, j,$ *and* $k$ will be $1, 2, \cdots, n$.

**Note:** Since we discuss a system of quasilinear equations (1), we choose a solution $\boldsymbol{u}(\boldsymbol{x}, t)$ on the domain $D$ of $(\boldsymbol{x}, t)$-space and all results will be valid locally for the given solution $\boldsymbol{u}(\boldsymbol{x}, t)$.

**Definition 1.** We define the system (1) as hyperbolic in $D$ with $t$ as time-like variable if, given an arbitrary unit vector $\boldsymbol{n}$, the characteristic equation

$$Q(\boldsymbol{x}, t, \boldsymbol{u}, \boldsymbol{n}) \equiv \det \left[ n_\alpha B^{(\alpha)} - cA \right] = 0 \tag{2}$$

has $n$ real roots (called eigenvalues) and eigenspace is complete at each point of $D$.

We denote the eigenvalues as

$$c_1, c_2, \ldots, c_n \tag{3}$$

and left and right eigenvectors by $\boldsymbol{\ell}^{(i)}$ and $\boldsymbol{r}^{(i)}$, which satisfy

$$\boldsymbol{\ell}^{(i)}\ (n_\alpha B^{(\alpha)}) = c_i \boldsymbol{\ell}^{(i)} A, \quad (n_\alpha B^{(\alpha)})\boldsymbol{r}^{(i)} = c_i A \boldsymbol{r}^{(i)}. \tag{4}$$

Suppose an eigenvalue $c_i(\boldsymbol{x}, t, \boldsymbol{u}, \boldsymbol{n})$ is repeated $p_i$ times in the set (3), completeness of eigenspace at each point of $D$ implies that the number of linearly independent left eigenvectors (and hence also right eigenvectors) corresponding to $c_i$ is $p_i$.

**Definition 2.** Any quantity associated with the $i$th eigenvalue $c_i$ will be referred to as the quantity belonging to $i$th or $c_i$ characteristic field.

**Eikonal Equation for a Wavefront:** Let $\Omega : \varphi(\boldsymbol{x}, t) = \alpha$, $\alpha =$ constant be a one parameter family of characteristic surfaces in a characteristic field of an eigenvalue $c$ of the system (1). Since

$$n_\alpha = \varphi_{x_\alpha}/|\boldsymbol{\nabla}\varphi|, \qquad \textit{and} \qquad c = -\varphi_t/|\boldsymbol{\nabla}\varphi| \tag{5}$$

characteristic equation (2) takes the form of a first order PDE, called eikonal equation,

$$Q(\boldsymbol{x}, t; \boldsymbol{\nabla}\varphi, \varphi_t) \equiv \det(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}) = 0. \tag{6}$$

- A surface $\Omega$ satisfying (6) is a characteristic manifold in space-time:

  $$\Omega := \varphi(\boldsymbol{x}, t) = 0. \tag{7}$$

  Its section by the plane $t = \textit{constant}$ is also represented by $\varphi(\boldsymbol{x}, t) = 0$, where $t =$ is kept constant.

- Projection of the section of $\Omega$ by $t = \textit{constant}$ on the physical space $\boldsymbol{x}$ is a wavefront. It is denoted by

  $$\Omega_t := \varphi(\boldsymbol{x}, t) = 0 \qquad t = \quad \textit{constant}. \tag{8}$$

  Unit normal $\boldsymbol{n}$ of the wavefront is given by

  $$\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}. \tag{9}$$

- Velocity of propagation of the wavefront (i.e. the normal velocity) is given by

  $$c = -\varphi_t/(|\nabla\varphi|). \tag{10}$$

When $|\boldsymbol{n}| = 1$ in (2), the eigenvalue $c$ is the **normal velocity** or simply **velocity** of a wavefront $\Omega_t : \varphi(\boldsymbol{x}, t) = \alpha$, ($t$ and $\alpha =$ constant) in $\boldsymbol{x}$-space.

**Another Form of the Eikonal Equation for a Wavefront Belonging to a Particular Characteristic Field** $c$**:** It is simple to show that $\varphi(\boldsymbol{x}, t)$ satisfies

$$\tilde{Q}(\boldsymbol{x}, t, \boldsymbol{\nabla}\varphi, \varphi_t) := (\boldsymbol{\ell}A\boldsymbol{r})\varphi_t + (\boldsymbol{\ell}B^{(\alpha)}\boldsymbol{r})\varphi_{x_\alpha} = 0. \tag{11}$$

We use this form (11) to prove

**The Bicharacteristic Theorem:** For the hyperbolic system (1) of $n$ first order PDEs we state the Bicharacteristic Theorem as:
The ray equations are

$$\frac{dx_\alpha}{dt} = \frac{\boldsymbol{\ell} B^{(\alpha)}\boldsymbol{r}}{\boldsymbol{\ell} A\boldsymbol{r}} \equiv \chi_\alpha \quad \textit{and} \tag{12}$$

$$\frac{dn_\alpha}{dt} = -\frac{\ell}{\boldsymbol{\ell} A\boldsymbol{r}}\boldsymbol{\ell}\left\{n_\beta\left(n_\gamma\frac{\partial B^{(\gamma)}}{\partial\eta^\alpha_\beta} - c\frac{\partial A}{\partial\eta^\alpha_\beta}\right)\right\}\boldsymbol{r} \equiv \psi_\alpha. \tag{13}$$

The transport equation along a ray is

$$\boldsymbol{\ell} A\frac{d\boldsymbol{u}}{dt} + \boldsymbol{\ell}(B^{(\alpha)} - \chi_\alpha A)\frac{\partial\boldsymbol{u}}{\partial x_\alpha} + \boldsymbol{\ell} C = 0. \tag{14}$$

In (13)

$$\frac{\partial}{\partial\eta^\alpha_\beta} = n_\beta\frac{\partial}{\partial x_\alpha} - n_\alpha\frac{\partial}{\partial x_\beta}. \tag{15}$$

This operator is tangential to the wavefront $\Omega_t$ which is projection of a section of the characteristic manifold $\Omega$ and hence also to $\Omega$.
The vector $\boldsymbol{\chi} = (\chi_1, \cdots, \chi_m)$ is called the ray velocity and the operator tangential to the characteristic manifold $\Omega$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_\alpha\frac{\partial}{\partial x_\alpha} \tag{16}$$

represents differentiation in a bicharacteristic direction.

**An interesting property of $\frac{\partial}{\partial\eta^\alpha_\beta}$ in (15):** Define

$$L_\alpha = n_\beta\frac{\partial}{\partial\eta^\alpha_\beta}, \qquad \alpha = 1, 2, \ldots, d. \tag{17}$$

The operator $L_\alpha$ can also be written in the form

$$L_\alpha = n_\beta\left(n_\beta\frac{\partial}{\partial x_\alpha} - n_\alpha\frac{\partial}{\partial x_\beta}\right) = \frac{\partial}{\partial x_\alpha} - n_\alpha\left(n_\beta\frac{\partial}{\partial x_\beta}\right), \ \textit{i.e.} \tag{18}$$

$$\boldsymbol{L} = \boldsymbol{\nabla} - \boldsymbol{n}\langle\boldsymbol{n}, \boldsymbol{\nabla}\rangle. \tag{19}$$

Thus we see a beatiful result "$\boldsymbol{L}$ is obtained from $\boldsymbol{\nabla}$ by subtracting from $\boldsymbol{\nabla}$ its component normal to the wavefront $\Omega_t$".