Contemporary Mathematics
Volume **681**, 2017
http://dx.doi.org/10.1090/conm/681/13687

Lower-dimensional Fefferman measures via the Bergman kernel

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Abstract. Motivated by the theory of Hausdorff measures, we propose a new construction of the Fefferman hypersurface measure. This construction reveals the existence of non-trivial Fefferman-type measures on the boundary of some domains — such as products of balls — which are outside the purview of Fefferman's original definition. We also show that these measures enjoy certain transformation properties under biholomorphic mappings.

1. Introduction

In his paper *Parabolic invariant theory in complex analysis* (see [**7**]), Fefferman observed that a certain positive $(2d-1)$-form, σ_Ω, on the boundary of a $\mathcal{C}^2$-smooth strongly pseudoconvex domain, $\Omega \subset \mathbb{C}^d$, satisfies the following transformation law:

$$F^*\sigma_{F(\Omega)} = |\det \mathrm{J}_{\mathbb{C}}F|^{\frac{2d}{d+1}}\sigma_\Omega, \tag{1.1}$$

where F is a biholomorphism on Ω that is $\mathcal{C}^2$-smooth on $\overline{\Omega}$. This form σ_Ω, or the *Fefferman hypersurface measure on* $\partial\Omega$, is defined (up to a constant) by

$$\sigma_\Omega \wedge d\rho = 4^{\frac{d}{d+1}}\mathscr{M}(\rho)^{\frac{1}{d+1}}\omega_{\mathbb{C}^d}, \tag{1.2}$$

where $\omega_{\mathbb{C}^d}$ is the standard volume form on $\mathbb{C}^d$, ρ is a defining function for Ω with $\Omega = \{\rho < 0\}$, and

$$\mathscr{M}(\rho) = -\det\begin{pmatrix}\rho & \rho_{\overline{z_k}}\\ \rho_{z_j} & \rho_{z_j\overline{z_k}}\end{pmatrix}_{1\le j,k\le d}.$$

Our interest in this measure arises from (1.1), and more specifically, its invariance under volume-preserving biholomorphisms. In view of this property, this measure has been used to study Szegő projections on CR-manifolds ([**11**]), volume-preserving CR invariants, isoperimetric problems (see [**10**] and [**3**]) and invariant metrics ([**4**]). The standard Euclidean surface area measure notably lacks such a transformation law when $d > 1$.

As strong pseudoconvexity is a biholomorphically invariant version of strong convexity (see Definition 2.1), it is natural to ask whether an analogue of Fefferman's measure exists in the affine setting. In 1923, Blaschke ([**5**]) observed that if $D \subset \mathbb{R}^d$

2010 *Mathematics Subject Classification.* Primary 32T15, 52A23, 28A78.

This work was partially supported by the NSF under grant numbers DMS 1161735 and DMS 1500142.

is a $\mathcal{C}^2$-smooth convex body, the *Blaschke affine surface area measure on ∂D* given by

$$\mu_D = \kappa^{\frac{1}{d+1}} s_{\text{Euc}},$$

where κ and s_{Euc} are the Gaussian curvature function and the Euclidean surface area form on ∂D, respectively, obeys the following identity:

$$A^* \mu_{A(D)} = |\det \mathrm{J}_{\mathbb{R}} A|^{\frac{d-1}{d+1}} \mu_D,$$

where A is an affine transformation of $\mathbb{R}^d$. In particular, μ_D is invariant under equi-affine (volume-preserving affine) maps. This initiated a project of characterizing Blaschke's measure in ways that did not rely on the smoothness of the convex body in question (see [**8**, Chap. 1.10] and [**13**] for details). Many of these methods utilize a volume-approximation approach — elucidated below by two results, chosen specifically due to their influence on the main ideas of this article.

RESULT 1.1 (Schütt-Werner, [**15**]). *Let $D \Subset \mathbb{R}^d$ be a convex domain. For any $\delta > 0$, let D_δ denote the intersection of all the halfspaces in $\mathbb{R}^d$ whose hyperplanes cut off a set of volume δ from D. Then,*

$$\lim_{\delta \to 0} \frac{\text{vol}(D) - \text{vol}(D_\delta)}{\delta^{\frac{2}{d+1}}}$$

exists, and coincides (up to a dimensional constant) with the total Blaschke affine surface area measure of ∂D when D is $\mathcal{C}^2$-smooth.

RESULT 1.2 (Ludwig, [**14**]). *Let $D \Subset \mathbb{R}^d$ be a $\mathcal{C}^2$-smooth strongly convex domain. For $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the set of all d-dimensional convex polyhedra with at most n facets. Then,*

$$\inf\{\text{vol}(D \Delta P) : P \in \mathcal{P}_n\} \sim \ell_1 \left(\int_{\partial D} \mu_D \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.3}$$

as $n \to \infty$, where Δ denotes the symmetric difference between sets and ℓ_1 is a dimensional constant.

In order to establish an analogous project for the Fefferman hypersurface measure, results 1.1 and 1.2 have been generalized to the holomorphic setting (see [**1**] and [**9**]), thus providing new characterizations of σ_Ω. We now paraphrase one particularly relevant result of that kind.

RESULT 1.3 (Gupta, [**9**]). *Let $\Omega \Subset \mathbb{C}^2$ be a $\mathcal{C}^\infty$-smooth strongly pseudoconvex domain and K_Ω be its Bergman kernel (defined in Section 2). For $n \in \mathbb{N}$, let $\mathcal{BP}_n$ denote the collection of all relatively compact sets in Ω of the form*

$$P = \left\{ z \in \Omega : |K_\Omega(w^j, z)| < m_j, j = 1, ..., n \right\},$$

where, $w^1, ..., w^n \in \partial\Omega$ and $m_1, ..., m_n > 0$. Then,

$$\inf\{\text{vol}(\Omega \setminus P) : P \in \mathcal{BP}_n\} \sim \ell_2 \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}} \tag{1.4}$$

as $n \to \infty$, where ℓ_2 is a constant independent of Ω.

As neither smoothness nor strong pseudoconvexity are needed to define the Bergman kernel, we ask whether the procedure outlined in Result 1.3 can be used to define the Fefferman hypersurface measure for more general domains. As an example, we consider the unit bidisc $\mathbb{D}^2$ and observe that

$$\lim_{n\to\infty} \sqrt{n}\ \inf\{\mathrm{vol}(\mathbb{D}^2 \setminus P) : P \in \mathcal{BP}_n\} = 0 \qquad \text{as } n \to \infty.$$

This is hardly surprising since $\sigma_{\mathbb{D}^2}$ makes sense and vanishes almost everywhere on the boundary of $\mathbb{D}^2$ (see (1.2)). On further inspection, we find that

$$\lim_{n\to\infty} \frac{n}{\log n}\ \inf\{\mathrm{vol}(\mathbb{D}^2 \setminus P) : P \in \mathcal{BP}_n\} = \ell, \quad \text{for some } \ell \neq 0. \tag{1.5}$$

With (1.4) in mind, we ask whether there is some measure on $\partial\mathbb{D}^2$ — possibly supported on a proper subset — that determines the limit ℓ in (1.5). Given the invariance properties of the left-hand side in (1.5), a good choice would be the standard product measure on the distinguished boundary, $\partial\mathbb{D} \times \partial\mathbb{D}$, of $\mathbb{D}^2$ — a measure that is conventionally used to set up Hardy spaces on the bidisc. This motivates the following

QUESTION. Is there a unified construction of boundary measures which are invariant under volume-preserving biholomorphisms, that yields σ_Ω for strongly pseudoconvex domains and the measure discussed above for the bidisc?

In this article, we answer the above question in the affirmative. Our construction is motivated by Result 1.3, but replaces the full Bergman kernel with the arguably simpler diagonal Bergman kernel.

DEFINITION 1.4. Let $\Omega \subset \mathbb{C}^d$ be a bounded domain, K_Ω be its Bergman kernel, and $\omega_{\mathbb{C}^d}$ be viewed as a measure on $\overline{\Omega}$. We set, for any $M > 0$,

$$\Omega_M := \{z \in \Omega : K_\Omega(z, z) > M\}.$$

The *Hausdorff-Fefferman measure* on $\partial\Omega$ is defined as

$$\widetilde{\sigma}_\Omega(A) := \text{ weak-}* \text{ limit of } \frac{1}{\mathrm{vol}(\Omega_M)}\chi_{\Omega_M}\omega_{\mathbb{C}^d} \text{ as } M \to \infty,$$

when it exists, where χ_A denotes the indicator function of A.

We will later encounter Definition 4.2 which is a slight generalization of Definition 1.4. Under certain restrictions on the domain Ω, $\tilde{\sigma}_\Omega$ does exist, and expands the scope of Fefferman's original definition as can be seen from the following result (proved in Section 4):

PROPOSITION 1.5. *Here $\approx$ denotes equality up to renormalizations, $\mathbb{B}^d$ denotes the unit ball in $\mathbb{C}^d$, and all volume and hypersurface forms are viewed as measures.*

(1) *If $\Omega \Subset \mathbb{C}^d$ is a strongly pseudoconvex domain, then $\widetilde{\sigma}_\Omega \approx \sigma_\Omega$.*
(2) *If $\Omega = \mathbb{B}^d \times \mathbb{B}^d$, then $\widetilde{\sigma}_\Omega$ is supported on $\partial\mathbb{B}^d \times \partial\mathbb{B}^d$ and $\widetilde{\sigma}_\Omega \approx s_{\mathbb{B}^d} \cdot s_{\mathbb{B}^d}$, where $s_{\mathbb{B}^d}$ is the standard surface area on $\partial\mathbb{B}^d$.*
(3) *If $\Omega = \mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$, with $d_1 > d_2$, then $\widetilde{\sigma}_{\partial\Omega}$ is supported on $\partial\mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$ and $\widetilde{\sigma}_\Omega \approx h \cdot s_{\mathbb{B}^{d_1}} \cdot \omega_{\mathbb{C}^{d_2}}$, where $h(z, w) = K_{\mathbb{B}^{d_2}}(z, w)^{1/(d_1+1)}$ and K_Ω is the Bergman kernel of Ω.*

From our vantage point, it is crucial that the transformation and invariance properties of the Fefferman hypersurface measure are inherited by the new measure $\tilde{\sigma}_\Omega$. This is true under certain conditions, as is seen in our second result, for which we need the following definition:

Definition 1.6. The *Hausdorff-Fefferman dimension* of a bounded domain $\Omega \subset \mathbb{C}^d$ is said to exist if

$$\sup\{\alpha > 0 : \liminf_{M\to\infty} M^{\frac{1}{\alpha}} \operatorname{vol}(\Omega_M) = \infty\} = \inf\{\alpha > 0 : \limsup_{M\to\infty} M^{\frac{1}{\alpha}} \operatorname{vol}(\Omega_M) = 0\}. \tag{1.6}$$

In this case, we denote the above quantity by $\dim_{\mathrm{HF}}(\Omega)$.

Remarks 1.7. Hereafter, we use the notation $\dim_{\mathrm{HF}}(\Omega)$ under the implicit assumption that the Hausdorff-Fefferman dimension of Ω exists. By definition, this quantity is positive and finite.

We are now in the position to state a transformation law for $\tilde{\sigma}_\Omega$.

Theorem 1.8. *Let $\Omega^1, \Omega^2 \Subset \mathbb{C}^d$ be domains, and $F : \Omega^1 \to \Omega^2$ a biholomorphism such that $F \in \mathcal{C}^1(\overline{\Omega^1})$ and $\mathrm{J}_{\mathbb{C}}F$ is non-vanishing. Suppose*

(*i*) $\alpha := \dim_{\mathrm{HF}}(\Omega^1) < \infty$;
(*ii*) *for $a > 0$, $\operatorname{vol}(\Omega^1_M)/\operatorname{vol}(\Omega^1_{aM})$ has a limit in $[0,\infty]$ as $M \to \infty$;*
(*iii*) $\operatorname{vol}(\Omega^1_M)/\operatorname{vol}(\Omega^2_M)$ *has a limit in $[0,\infty]$ as $M \to \infty$.*

Then,

$$F^*\tilde{\sigma}_{\Omega^2} \approx |\det \mathrm{J}_{\mathbb{C}}F|^{2\left(1-\frac{1}{\alpha}\right)}\tilde{\sigma}_{\Omega^1}, \tag{1.7}$$

where $\approx$ denotes equality up to renormalizations as probability measures.

This article is organized as follows. We give some notation and definitions in Section 2. In section 3, we will motivate the Hausdorff-Fefferman dimension, which not only plays an integral role in the transformation law given by Theorem 1.8, but also offers an invariant of independent geometric interest. We expand on the construction of $\tilde{\sigma}_\Omega$ and give the proofs of our results in Section 4.

2. Definitions

In this article, $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$ and $\mathbb{B}^d$ denotes the unit ball in $\mathbb{C}^d$. For $D \subseteq \mathbb{R}^n$, $\mathcal{C}(D)$ is the set of all continuous functions on D, and $\mathcal{C}^k(D)$, $k \geq 1$, denotes the set of all functions that are k-times continuously differentiable in some open neighborhood of D. For a domain $\Omega \subset \mathbb{C}^d$, $\mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω. When well defined, $\mathrm{J}_{\mathbb{R}}f(x)$ and $\mathrm{J}_{\mathbb{C}}f(x)$ denote the real and complex Jacobian matrix, respectively, of f at x. For any Lebesgue measurable set $D \subset \mathbb{C}^d$, $\operatorname{vol}(D)$ denotes its total Lebesgue measure.

In our analogy between convex and complex analysis, the role of convexity is played by pseudoconvexity:

Definition 2.1. A $\mathcal{C}^2$-smooth domain $\Omega \subset \mathbb{C}^d$ is called *strongly pseudoconvex* if there is a $\mathcal{C}^2$-smooth function ρ defined in a neighborhood U of $\overline{\Omega}$ such that $\Omega = \{z \in U : \rho(z) < 0\}$, and for every $z \in \partial\Omega$,

$$\sum_{1\leq j,k\leq d} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z) v_j \overline{v_k} > 0 \tag{2.1}$$

for all $v = (v_1, ..., v_d) \in \mathbb{C}^d \setminus \{0\}$ satisfying $\sum_{j=1}^d \frac{\partial \rho}{\partial z_j}(z) v_j = 0$.

A (possibly non-smooth) domain $\Omega \subset \mathbb{C}^d$ is called *pseudoconvex* if it can be exhausted by strongly pseudoconvex domains, i.e, $\Omega = \cup_{j\in\mathbb{R}}\Omega_j$ with each Ω_j strongly pseudoconvex and $\Omega_j \subseteq \Omega_k$ for $j < k$.

Although we are motivated by methods in convex analysis, our approach is novel in its use of the following complex-analytic tool.

DEFINITION 2.2. The *Bergman kernel of a domain* Ω, $K_\Omega : \Omega \times \Omega \to \mathbb{C}$, is the reproducing kernel of the Hilbert space $\{f \in \mathcal{H}(\Omega) : ||f||_2 < \infty\}$, where $||f||_2$ is the L^2-norm of f with respect to the Lebesgue measure on Ω.

We will abbreviate $K_\Omega(z,z)$ to $K_\Omega(z)$. The Bergman kernel displays many interesting and important properties (see [**6**] for a survey), the most important one for our purpose being the following:

FACT. If $F : \Omega_1 \to \Omega_2$ is a biholomorphism between bounded domains in $\mathbb{C}^d$. Then,

$$\det \mathrm{J}_{\mathbb{C}} F(z) \cdot K_{\Omega_2}(F(z), F(w)) \cdot \overline{\det \mathrm{J}_{\mathbb{C}} F(w)} = K_{\Omega_1}(z,w),$$

for all $z, w \in \Omega_1$.

We follow standard terminology and call a domain $\Omega \Subset \mathbb{C}^d$ *Bergman exhaustive* if for every $w \in \partial\Omega$, $\lim_{z\to w} K_\Omega(z) = \infty$.

3. The Hausdorff-Fefferman dimension

We begin this section by illustrating the relevance of the exponents of the Jacobian terms in the transformation identities (1.1) and (1.7). Following the exposition in [**2**, Section 2], we consider the $\mathbb{C}$-bundles $\mathcal{O}(j,k)$ over the projective space $\mathbb{CP}^d$. Any section of $\mathcal{O}(j,k)$ over a subset $E \subset \mathbb{CP}^d$ is given by a $\mathbb{C}$-valued function G on the corresponding dilation-invariant subset of $\mathbb{C}^{d+1} \setminus \{0\}$ satisfying the homogeneity condition

$$G(\lambda z) = \lambda^j \overline{\lambda^k} G(z).$$

Here $j, k \in \mathbb{R}$ with $j - k \in \mathbb{Z}$. The space of continuous sections of $\mathcal{O}(j,k)$ over E is denoted by $\Gamma(E; j, k)$. Owing to Remark 1 in [**2**, Section 2] we are allowed to use the notation

$$g(z_1, ..., z_d)(dz_1 \wedge \cdots \wedge dz_d)^{\frac{-j}{d+1}} (d\overline{z_1} \wedge \cdots \wedge d\overline{z_d})^{\frac{-k}{d+1}}$$

for sections of $\mathcal{O}(j,k)$. Now, let $\mathscr{S}$ denote a biholomorphically invariant collection of CR-manifolds in $\mathbb{C}^d$ such that for each $S \in \mathscr{S}$, there is a finite positive measure σ_S such that

$$F^* \nu_{F(S)} = |\det \mathrm{J}_{\mathbb{C}} F|^{2\beta} \nu_S$$

for any biholomorphism F in a neighborhood of S, where $\beta > 0$ does not depend on S. Then for any submanifold $C \subset \mathbb{CP}^d$ that restricts to an element in $\mathscr{S}$ in each affine chart, there is an $\mathcal{O}(\beta(d+1), \beta(d+1))$-valued measure ν given in the affine chart $U_0 = \{[z_0 : \cdots : z_d] \in \mathbb{CP}^d : z_0 \neq 0\}$ by

$$\nu := \frac{\nu_{C\cap U_0}}{(dz_1 \wedge \cdots \wedge dz_d)^\beta (d\overline{z_1} \wedge \cdots \wedge d\overline{z_d})^\beta}.$$

This allows us to define the L^2-norm

$$||G||_C^2 = \int_C G\overline{G} d\nu$$

for $G \in \Gamma(C; j, k)$ with $j + k = \beta(d+1)$. Thus, the quantity β plays a role in setting up appropriate Hardy spaces in the projective space. This has been done in [**2**, Section 8.] for the case where $\mathscr{S}$ is the collection of all smooth strongly pseudoconvex hypersurfaces in $\mathbb{C}^d$, with $\beta = \frac{d}{d+1}$. The exponent of the Jacobian term in (1.1) also plays a role in designing constant-Jacobian biholomorphic invariants such as the isoperimtric quotient in [**3**]. As this exponent can be deduced from the Hausdorff-Fefferman dimension of the domain in question (see Definition 1.6), we devote the rest of this section to some basic properties of $\dim_{\mathrm{HF}}$.

PROPOSITION 3.1. *Let $F : \Omega^1 \to \Omega^2$ be a biholomorphism such that $a \leq |\det \mathrm{J}_{\mathbb{C}} F| \leq b$ for some $a, b > 0$. If Ω^1 admits a Hausdorff-Fefferman dimension, then so does Ω^2, and* $\dim_{\mathrm{HF}}(\Omega^2) = \dim_{\mathrm{HF}}(\Omega^1)$.

PROOF. Let $K_j(z) := K_{\Omega^j}(z)$ for $z \in \Omega^j$, $j = 1, 2$. Observe that

$$\begin{aligned} F^{-1}(\Omega^2_M) &= \{z \in \Omega^1 : K_2(F(z)) > M\} \\ &= \{z \in \Omega^1 : K_1(z) > M|\det \mathrm{J}_{\mathbb{C}} F(z)|^2\} \\ &\subseteq \{z \in \Omega^1 : K_1(z) > Ma^2\}. \end{aligned} \tag{3.1}$$

Therefore,

$$\begin{aligned} \mathrm{vol}(\Omega^2_M) &= \int_{F^{-1}(\Omega^2_M)} |\det \mathrm{J}_{\mathbb{C}} F(z)|^2 \, \omega_{\mathbb{C}^d}(z) \\ &\leq \int_{\Omega^1_{Ma^2}} |\det \mathrm{J}_{\mathbb{C}} F(z)|^2 \, \omega_{\mathbb{C}^d}(z) \leq b^2 \, \mathrm{vol}(\Omega^1_{Ma^2}). \end{aligned} \tag{3.2}$$

As a and b are independent of M, we get that $\dim_{\mathrm{HF}}(\Omega^2) \leq \dim_{\mathrm{HF}}(\Omega^1)$. The reverse inequality also holds as $F^{-1} : \Omega^2 \to \Omega^1$ satisfies the hypothesis of the claim. □

COROLLARY 3.2. *The Hausdorff-Fefferman dimension of a domain is invariant under volume-preserving biholomorphisms.*

We now use known estimates and formulas for the Bergman kernel to compute the Hausdorff-Fefferman dimensions of two types of examples — smooth (with some strong pseudoconvexity assumption), and non-smooth (with a product structure), starting with some preliminary estimates on $\dim_{\mathrm{HF}}$.

LEMMA 3.3. *Let $\Omega \Subset \mathbb{C}^d$ be a $\mathcal{C}^1$-smooth domain. Then,* $\dim_{\mathrm{HF}}(\Omega) \leq d + 1$.

PROOF. Let $z \in \Omega$ and $\mathrm{dist}(z, \partial\Omega)$ denote the Euclidean distance of z from $\partial\Omega$. This proof relies on the well-known inequality

$$K_\Omega(z) \leq \frac{\text{const.}}{\mathrm{dist}(z, \partial\Omega)^{d+1}}, \text{ for all } z \in \Omega,$$

which is obtained by rolling a ball of fixed radius in Ω along $\partial\Omega$. Thus, $\{z \in \Omega : K_\Omega(z) > M\} \subseteq \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < (\text{const.})M^{1/(d+1)}\}$. The regularity assumption on Ω yields

$$\mathrm{vol}(\Omega_M) \leq \mathrm{vol}\{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < (\text{const.})M^{1/(d+1)}\} \sim \frac{1}{M^{1/(d+1)}} \quad \text{as } M \to \infty.$$

Hence, the claim. □

LEMMA 3.4. *Let $\Omega^j \Subset \mathbb{C}^{d_j}$, $j = 1, \ldots, k$, be Bergman exhaustive domains. Then,* $\dim_{\mathrm{HF}}(\Omega^1 \times \cdots \times \Omega^k) \geq \max\{\dim_{\mathrm{HF}}(\Omega^j) : 1 \leq j \leq k\}$.

PROOF. Let $k = 2$. It is known that $K_{\Omega^1 \times \Omega^2}\big((z, w)\big) = K_{\Omega^1}(z) K_{\Omega^2}(w)$. Hence,

$$(\Omega^1 \times \Omega^2)_M = \bigcup_{w \in \Omega^2} \left\{(z, w) : z \in \Omega^1_{M/K_{\Omega_2}(w)}\right\} \supset \bigcup_{w \in \Omega^2} \left\{(z, w) : z \in \Omega^1_{M/k_2}\right\},$$

where $k_2 := \min_{w\in\overline{\Omega^2}} K_{\Omega_2}(w)$. Thus, for all $\alpha > 0$,

$$M^{1/\alpha}\operatorname{vol}\left((\Omega^1\times\Omega^2)_M\right) \geq M^{1/\alpha}\operatorname{vol}\left(\Omega^2\right)\operatorname{vol}\left(\Omega^1_{M/k_2}\right).$$

As k_2 and $\operatorname{vol}(\Omega^2)$ are independent of M,

$$\left\{\alpha : \limsup_{M\to\infty} M^{\frac{1}{\alpha}}\operatorname{vol}\left((\Omega^1\times\Omega^2)_M\right)=0\right\} \subseteq \left\{\alpha : \limsup_{M\to\infty} M^{\frac{1}{\alpha}}\operatorname{vol}(\Omega^1_M)=0\right\}.$$

Repeating the argument with Ω^2_M instead, we get that $\dim_{\mathrm{HF}}(\Omega^1\times\Omega^2) \geq \max\{\dim_{\mathrm{HF}}(\Omega^j) : j=1,2\}$. The argument for general $k\in\mathbb{N}_+$ follows from the fact that if $\Omega^1,\ldots,\Omega^k$ satisfy the hypothesis of the proposition, then so do $\Omega^1\times\cdots\times\Omega^{k-1}$ and Ω^k. □

PROPOSITION 3.5. *(a) Let $\Omega \Subset \mathbb{C}^d$ be a $\mathcal{C}^1$-smooth domain. Suppose $\overline{\partial} : L^2_{(0,0)}(\Omega)\to L^2_{(0,1)}(\Omega)$ has closed range, and there is a $p\in\partial\Omega$ such that $\partial\Omega$ is $\mathcal{C}^2$-smooth and strongly pseudoconvex in a neighborhood of p. Then,* $\dim_{\mathrm{HF}}(\Omega)=d+1$.
(b) Let $\Omega\Subset\mathbb{C}^k$ be a Bergman exhaustive domain such that

$$\int_{\Omega\setminus\Omega_M}\sqrt{K_\Omega(z)}\,\omega_{\mathbb{C}^d}(z) = o(M^\eta) \text{ as } M\to\infty, \text{ for every } \eta>0. \tag{3.3}$$

Then, $\dim_{\mathrm{HF}}(\mathbb{B}^d\times\Omega)=\max\{d+1,\dim_{\mathrm{HF}}(\Omega)\}$.

REMARK 3.6. An elementary example of a domain that satisfies condition (3.3) is $\mathbb{B}^d$, $d\geq 1$. Moreover, if $\Omega^j\subset\mathbb{C}^{d_j}$, $j=1,\ldots,k$, are domains that satisfy the hypotheses of (b) in Propostion 3.5, then so does $\Omega^1\times\cdots\times\Omega^k$. Thus, in particular, $\dim_{\mathrm{HF}}(\mathbb{B}^{d_1}\times\cdots\mathbb{B}^{d_l}) = \max_{1\leq j\leq l}\{d_j+1\}$.

PROOF OF PROPOSITION 3.5. (a) As proved in Proposition 3.3, $\dim_{\mathrm{HF}}(\Omega)\leq d+1$. By Hörmander's theorem on the boundary behavior of the (diagonal) Bergman kernel (see Theorem 3.5.1 in [**12**]), there exists a neigborhood $U\subset\partial\Omega$ of p and a continuous positive function $f:U\to\mathbb{R}$ such that

$$\operatorname{dist}(z,\partial\Omega)^{d+1}K_\Omega(z)\to f(z_0),\quad z\to z_0\in U.$$

Thus, for any $V\Subset U$, there is a $c>0$, such that $\{z\in\Omega : K_\Omega(z)>M\}\supseteq\{z\in\Omega : \operatorname{dist}(z,V)<cM^{1/(d+1)}\}$. We get,

$$\operatorname{vol}(\Omega_M)\geq\frac{c's(V)}{M^{1/(d+1)}},$$

where $c'>0$ is a constant and $s(V)$ is the Euclidean surface area of V. This gives the required lower bound on $\dim_{\mathrm{HF}}(\Omega)$.

(b) We observe that for $\mathfrak{b}_d=\operatorname{vol}(\mathbb{B}^d)$,

$$K_{\mathbb{B}^d\times\Omega}\left((z,w)\right)=\frac{1}{\mathfrak{b}_d(1-||z||^2)^{d+1}}K_\Omega(w).$$

Thus, we may write

$$\left(\mathbb{B}^d\times\Omega\right)_M=\{(z,w): z\in\mathbb{B}^d, w\in\Omega_{M\mathfrak{b}_d}\}\bigcup\{(z,w): z\in(\mathbb{B}^d)_{M/K_\Omega(w)}, w\in\Omega\setminus\Omega_{M\mathfrak{b}_d}\}. \tag{3.4}$$

Now, fix an $\alpha>\max\{d+1,\dim_{\mathrm{HF}}(\Omega)\}$ and let $\eta=\frac{1}{d+1}-\frac{1}{\alpha}$. Then, by the definition of $\dim_{\mathrm{HF}}$ and the hypothesis on Ω, given $\varepsilon>0$, there is an $M_\varepsilon>0$ such that $\operatorname{vol}(\Omega_{M\mathfrak{b}_d}) < \varepsilon M^{-1/\alpha}$ and $\int_{\Omega\setminus\Omega_{M\mathfrak{b}_d}} K_\Omega(w)^{1/(d+1)}\,\omega_{\mathbb{C}^d}(w) \leq \int_{\Omega\setminus\Omega_{M\mathfrak{b}_d}}\sqrt{K_\Omega(w)}\,\omega_{\mathbb{C}^d}(w) < \varepsilon M^\eta$, for all $M\geq M_\varepsilon$. Using the decomposition in (3.4) and the fact that $\mathrm{vol}(\mathbb{B}^d_M) \leq C_d/M^{1/(d+1)}$ for some dimensional constant C_d, we get

$$\begin{aligned}\mathrm{vol}\left((\mathbb{B}^d \times \Omega)_M\right) &= \mathrm{vol}(\mathbb{B}^d)\,\mathrm{vol}(\Omega_{M\mathfrak{b}_d}) + \int_{\Omega\setminus\Omega_{M\mathfrak{b}_d}} \mathrm{vol}\left(\mathbb{B}^d_{M/K_\Omega(w)}\right)\,\omega_{\mathbb{C}^d}(w)\\ &\leq \mathrm{vol}(\mathbb{B}^d)\frac{\varepsilon}{M^{1/\alpha}} + \frac{C_d}{M^{1/(d+1)}}\int_{\Omega\setminus\Omega_{M\mathfrak{b}_d}} K_\Omega(w)^{1/(d+1)}\,\omega_{\mathbb{C}^d}(w)\\ &< \left(\mathrm{vol}(\mathbb{B}^d) + C_d\right)\frac{\varepsilon}{M^{1/\alpha}},\end{aligned}$$

for $M \geq M_\varepsilon$. Thus, $\dim_{\mathrm{HF}}(\mathbb{B}^d \times \Omega) \leq \alpha$ for all $\alpha > \max\{d+1, \dim_{\mathrm{HF}}(\Omega)\}$. The lower bound follows from Proposition 3.4. □

REMARK 3.7. To see how the Hausdorff-Fefferman dimension distinguishes domains within a fixed ambient space, we observe that in $\mathbb{C}^3$, $\mathbb{B}^3$, $\mathbb{B}^1 \times \mathbb{B}^2$ and $\mathbb{B}^1 \times \mathbb{B}^1 \times \mathbb{B}^1$ have Hausdorff-Fefferman dimensions 4, 3 and 2, respectively.

4. Hausdorff-Fefferman measures

In analogy with Hausdorff measures, we would like to use the Hausdorff-Fefferman dimension of Ω to construct Fefferman-type measures on $\partial\Omega$. Under such a scheme, the total measure of $\partial\Omega$ would be $\lim_{M\to\infty} M^{1/\dim_{\mathrm{HF}}(\Omega)}\,\mathrm{vol}(\Omega_M)$. But, if we consider the simple example of $\Omega = \mathbb{D}\times\mathbb{D}$, we find that $\lim_{M\to\infty} M^{1/2}\,\mathrm{vol}(\Omega_M) = \infty$. Infact, $\mathrm{vol}(\Omega_M) \sim M^{-1/2}\log(M)$ as $M \to \infty$. In view of this logarithmic term, we expand the notion of the Hausdorff-Fefferman dimension in the following manner.

DEFINITION 4.1. Let $\Omega \subset \mathbb{C}^d$ be a bounded domain. Any increasing $d_\Omega \in \mathcal{C}((0,\infty))$ is called a *Hausdorff-Fefferman gauge function (or an* HF*-gauge function) of* Ω if

$$\lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M) \text{ exists, and is positive and finite.}$$

DEFINITION 4.2. Let Ω and d_Ω be as in Definition 4.1, and $\omega_{\mathbb{C}^d}$ be viewed as a measure on $\overline{\Omega}$. The *Hausdorff-Fefferman measure* on $\partial\Omega$ (corresponding to d_Ω) is defined as

$$\widetilde{\sigma}_\Omega(A) := \text{ weak-}* \text{ limit of } d_\Omega(M)\chi_{\Omega_M}\omega_{\mathbb{C}^d} \text{ as } M \to \infty,$$

when it exists, where χ_A denotes the indicator function of A.

REMARKS. (1) The weak-$*$ limit above is in the space $C(\overline{\Omega})^*$ — the space of bounded linear functionals on $\mathcal{C}(\overline{\Omega})$. By the Riesz representation theorem, $\widetilde{\sigma}_\Omega$ is a finite, positive, regular, Borel measure on $\overline{\Omega}$ — in fact, the support of $\widetilde{\sigma}_\Omega$ is contained in $\partial\Omega$, but may be strictly smaller, as we see in Proposition 1.5.

(2) If we choose $d_\Omega(M) = \mathrm{vol}(\Omega_M)^{-1}$, we obtain the measure defined in Definition 1.4. When we leave $\tilde{\sigma}_\Omega$ unqualified, we are referring to this special choice of HF-gauge function.

It would be interesting to know which domains admit a Hausdorff-Fefferman measure. For now, we compute the examples stated in Proposition 1.5. We note that although the the result is for $\tilde{\sigma}_\Omega$ corresponding to $d_\Omega(M) = \mathrm{vol}(\Omega_M)^{-1}$, a different choice of HF-gauge function changes the resulting measure only up to a constant factor, and hence we do not place too much emphasis on the choice of d_Ω.

Proof of Proposition 1.5. (1) Let $\Omega \Subset \mathbb{C}^d$ be strongly pseudoconvex. As the range of $\overline{\partial} : L^2_{(0,0)}(\Omega) \to L^2_{(0,1)}(\Omega)$ is closed, we obtain by our computations in Proposition 3.5, that $d_\Omega(M) = M^{1/(d+1)}$ is an HF-gauge function for Ω. To compute $\widetilde{\sigma}_\Omega$ with respect to this $d_\Omega(M)$, we recall Hörmander's estimate:

$$\lim_{z \to z_0 \in \partial\Omega} r(z)^{d+1} K_\Omega(z) = \frac{d!}{\pi^d} \mathscr{M}(r)(z_0), \quad \forall z_0 \in \partial\Omega,$$

where r is a $\mathcal{C}^2$-smooth defining function for Ω, and $\mathscr{M}(r)$ is the Fefferman Monge-Ampére operator defined in Section 1. Thus, setting $n(z) := \left(\frac{\mathscr{M}(r)(z)}{\mathfrak{b}_d M}\right)^{\frac{1}{d+1}}$ and $\nu(z)$ to be the outward unit normal vector at $z \in \partial\Omega$, we have for any $f \in \mathcal{C}(\overline{\Omega})$, $\varepsilon > 0$, an M large enough so that

$$\{z - r\nu(z) \in \Omega : z \in \partial\Omega, r \in (0, n(z)(1-\varepsilon))\} \subseteq \Omega_M$$
$$\subseteq \{z - r\nu(z) \in \Omega : z \in \partial\Omega, r \in (0, n(z)(1+\varepsilon))\},$$

and

$$|f(z - r\nu(z)) - f(z)| < \varepsilon, \quad \forall z \in \partial\Omega, r \in [0, n(z)(1+\varepsilon)].$$

Therefore,

$$\begin{aligned} d_\Omega(M) \int_{\Omega_M} f\, \omega_{\mathbb{C}^d} &< M^{\frac{1}{d+1}} \int_{\partial\Omega} (f(z)+\varepsilon)(n(z)(1+\varepsilon))\, s_\Omega \\ &= (4^d \mathfrak{b}_d)^{\frac{-1}{d+1}} (1+\varepsilon) \int_{\partial\Omega} (f(z)+\varepsilon)\, \sigma_\Omega(z). \end{aligned}$$

Similarly, $d_\Omega(M) \int_{\Omega_M} f\, \omega_{\mathbb{C}^d} > (4^d \mathfrak{b}_d)^{\frac{-1}{d+1}} (1-\varepsilon) \int_{\partial\Omega} (f(z)-\varepsilon)\, \sigma_\Omega(z)$. Therefore,

$$\widetilde{\sigma}_\Omega \text{ (w.r.t. } d_\Omega) = (4^d \mathfrak{b}_d)^{\frac{-1}{d+1}} \sigma_\Omega \text{ (as measures)}.$$

Thus, after renormalizing both the measures, we obtain our claim.

(2) − (3) Let $\Omega = \mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$, $d_1 \geq d_2$. We set $K_{d_j} := K_{\mathbb{B}^{d_j}}$, $j = 1, 2$. We write

$$\mathrm{vol}(\Omega_M) = \mathfrak{T}_1 + \mathfrak{T}_2,$$

where

$$\begin{aligned} \mathfrak{T}_1 := \mathrm{vol}(\mathbb{B}^{d_1})\, \mathrm{vol}(\mathbb{B}^{d_2}_{M\mathfrak{b}_{d_1}}) &= \mathfrak{b}_{d_1}\mathfrak{b}_{d_2}\left(1 - \left(1 - (M\mathfrak{b}_{d_1}\mathfrak{b}_{d_2})^{-\frac{1}{(d_2+1)}}\right)^{d_2}\right) \\ &= \frac{d_2(\mathfrak{b}_{d_1}\mathfrak{b}_{d_2})^{d_2/(d_2+1)}}{M^{1/(d_2+1)}} + o(M^{-1/(d_2+1)}), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathfrak{T}_2 &:= \int_{\mathbb{B}^{d_2} \setminus \mathbb{B}^{d_2}_{M\mathfrak{b}_{d_1}}} \mathrm{vol}\left(\mathbb{B}^{d_1}_{M/K_{d_2}(w)}\right) \omega_{\mathbb{C}^d}(w) \\ &= \mathfrak{b}_{d_1} \int_{\mathbb{B}^{d_2} \setminus \mathbb{B}^{d_2}_{M\mathfrak{b}_{d_1}}} \left(1 - \left(1 - \left(\frac{K_{d_2}(w)}{\mathfrak{b}_{d_1} M}\right)^{\frac{1}{d_1+1}}\right)\right)^{d_1} \omega_{\mathbb{C}^d}(w) \\ &= \mathfrak{b}_{d_1} \sum_{r=1}^{d_1} (-1)^{r+1} \binom{d_1}{r} (\mathfrak{b}_{d_1} M)^{\frac{-r}{d_1+1}} I\left[M\mathfrak{b}_{d_1}; d_1; d_2; r\right] \end{aligned} \tag{4.2}$$

for

$$I[M; d_1; d_2; r] := \int_{\{w \in \mathbb{B}^{d_2} : K_{d_2}(w) \leq M\}} K_{d_2}(w)^{\frac{r}{d_1+1}}\, \omega_{\mathbb{C}^d}(w).$$

Now, writing out the expression for K_{d_2} and using polar co-ordinates, we have that

$$I[M;d_1;d_2;r] \;=\; \frac{d_2\mathfrak{b}_{d_2}}{(\mathfrak{b}_{d_2})^{\frac{r}{d_1+1}}}\beta\left[1-\frac{1}{(\mathfrak{b}_{d_2}M)^{1/(d_2+1)}};d_2,1-\frac{d_2+1}{d_1+1}r\right],$$

where $\beta[z;a,b]$ is the incomplete beta function $\int_0^z t^{a-1}(1-t)^{b-1}dt$. Since

$$\beta[1-x;a,b]=\begin{cases} C_{a,b}\; x^{-b}+o(x^{-b}), & \text{if } b<0;\\ \log\frac{1}{x}+C_a+O(x), & \text{if } b=0;\\ \beta(a,b)+O(x^b), & \text{if } 0<b<1,\end{cases}$$

as $x\to 0$, where $C_{a,b}, C_a>0$ are independent of x, we conclude that

$$(4.3)\quad I[M;d_1;d_2;r] = \begin{cases} \widetilde{C}_{d_1,d_2,r}\; M^{\frac{1-(d_2+1)r/(d_1+1)}{d_2+1}}+o(M^{1-\frac{d_2+1}{d_1+1}r}), & \text{if } \frac{d_2+1}{d_1+1}r>1;\\ \frac{d_2}{d_2+1}(\mathfrak{b}_{d_2})^{d_2/(d_2+1)}\log M+\widetilde{C}_{d_2}+O(M^{-1/(d_1+1)}), & \text{if } \frac{d_2+1}{d_1+1}r=1;\\ \frac{d_2\mathfrak{b}_{d_2}}{(\mathfrak{b}_{d_2})^{\frac{r}{d_1+1}}}\beta\left(d_2,1-\frac{d_2+1}{d_1+1}r\right)+O(M^{1-\frac{d_2+1}{d_1+1}r}), & \text{if } \frac{d_2+1}{d_1+1}r\in(0,1)\end{cases}$$

as $M\to\infty$, where $\widetilde{C}_{d_1,d_2,r}, \widetilde{C}_{d_2}>0$ are independent of M.

Our goal is to determine the asymptotic behavior of $\mathfrak{T}_2$ (see (4.2)), as $M\to\infty$.

Case i. $d_1=d_2$. We use (4.3) to note that

$$M^{-r/d_1+1}I\left[M\mathfrak{b}_{d_1};d_1;d_2;r\right]\sim\begin{cases} M^{-\frac{2r-1}{d_1+1}}=o(M^{-1/(d_1+1)}), & \text{if } r>1\\ M^{-\frac{1}{d_1+1}}\log M, & \text{if } r=1.\end{cases}$$

Combining this with (4.1) and (4.2), we get that $d_\Omega(M):=\frac{M^{1/(d+1)}}{\log(M)}$ is an HF-gauge function for $\Omega=\mathbb{B}^d\times\mathbb{B}^d$, and collecting the various constants,

$$(4.4)\qquad \lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M)=\frac{d^2}{d+1}(\mathfrak{b}_d)^{\frac{2d}{d+1}}.$$

Next, to compute $\widetilde{\sigma}_\Omega$ with respect to this d_Ω, let $\eta\in(0,1)$ and

$$\begin{aligned} R_\eta \;&:=\; \{(z,w)\in\mathbb{B}^d\times\mathbb{B}^d : \min\{||z||,||w||\}>\eta\};\\ |R|_{\eta,M} \;&:=\; \{(|z|,|w|)\in\mathbb{R}^2 : (z,w)\in\Omega_M\cap R_\eta\}.\end{aligned}$$

Due to rotational symmetry in each variable, $\mathrm{vol}(\Omega_M\cap R_\eta)=(2d\mathfrak{b}_d)^2\,\mathrm{vol}(|R|_{\eta,M})$. Now, for a fixed η, when $M>\mathfrak{b}_d^{-2}(1-\eta^2)^{-2d-2}$, it is easy to see that $\mathrm{vol}\,(\Omega_M\setminus R_\eta)\sim M^{\frac{-1}{d+1}}$ as $M\to\infty$. Therefore, for any $f\in\mathcal{C}(\overline{\Omega})$ and $\eta\in(0,1)$,

$$(4.5)\qquad \lim_{M\to\infty} d_\Omega(M)\int_{\Omega_M} f\;\omega_{\mathbb{C}^d}=\lim_{M\to\infty} d_\Omega(M)\int_{\Omega_M\cap R_\eta} f\;\omega_{\mathbb{C}^d}.$$

Next, fix an $\varepsilon>0$. Then, for η close enough to 1, we have that

$$|f(r\theta,r'\theta')-f(\theta,\theta')|<\varepsilon \quad \text{for any } r,r'\in(\eta,1) \text{ and } \theta,\theta'\in\partial\mathbb{B}^d.$$

Therefore,

$$\begin{aligned}\int_{\Omega_M\cap R_\eta} f\ \omega_{\mathbb{C}^d} &= \int_{|R|_{\eta,M}}\int_{\partial\mathbb{B}^d}\int_{\partial\mathbb{B}^d} f(r\theta, r'\theta')(rr')^{2d-1}s_{\mathbb{B}^d}(\theta)s_{\mathbb{B}^d}(\theta')drdr' \\ &< \left(\varepsilon + \int_{\partial\mathbb{B}^d\times\partial\mathbb{B}^d} f(\theta,\theta')s_{\mathbb{B}^d}(\theta)s_{\mathbb{B}^d}(\theta')\right)\mathrm{vol}(|R|_{\eta,M}) \\ &= \left(\int_{(\partial\mathbb{B}^d)^2} f\ s_{\mathbb{B}^d}s_{\mathbb{B}^d} + \varepsilon\right)\frac{\mathrm{vol}(\Omega_M\cap R_\eta)}{(2d\mathfrak{b}_d)^2}.\end{aligned}$$

Similarly,

$$\int_{\Omega_M\cap R_\eta} f\ \omega_{\mathbb{C}^d} \quad > \quad \left(\int_{(\partial\mathbb{B}^d)^2} f\ s_{\mathbb{B}^d}s_{\mathbb{B}^d} - \varepsilon\right)(1-\eta)^{2d-1}\frac{\mathrm{vol}(\Omega_M\cap R_\eta)}{(2d\mathfrak{b}_d)^2}.$$

Thus, combining (4.4) and (4.5), we get that as measures,

$$\begin{aligned}&\widetilde{\sigma}_{\mathbb{B}^d\times\mathbb{B}^d}\ (\text{w.r.t. } d_\Omega) \\ &\quad = \lim_{M\to\infty} d_\Omega(M)\frac{\mathrm{vol}(\Omega_M\cap R_\eta)}{(2d\mathfrak{b}_d)^2} = \frac{d^2}{d+1}\frac{(\mathfrak{b}_d)^{\frac{2d}{d+1}}}{(2d\mathfrak{b}_d)^2}s_{\mathbb{B}^d}s_{\mathbb{B}^d} = \frac{(\mathfrak{b}_d)^{\frac{-2}{d+1}}}{4(d+1)}s_{\mathbb{B}^d}s_{\mathbb{B}^d}.\end{aligned}$$

Case ii. $d_1 > d_2$. We divide the asymptotic behavior of $M^{-r/d_1+1}I\left[M\mathfrak{b}_{d_1}; d_1; d_2; r\right]$ into various cases to invoke (4.3), as in the previous case. At the end, we get that $\mathfrak{T}_2 \sim M^{-1/(d_1+1)}$ as $M \to \infty$. Combining this with (4.1) , we conclude that $d_\Omega(M) = M^{-1/(d_1+1)}$ acts as an HF-gauge function for $\Omega = \mathbb{B}^{d_1}\times\mathbb{B}^{d_2}$ as long as $d_2 < d_1$. Moreover,

$$\lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M) = d_1 d_2(\mathfrak{b}_{d_1})^{\frac{d_1}{d_1+1}}(\mathfrak{b}_{d_2})^{\frac{d_2}{d_2+1}}\beta\left(d_2, 1-\frac{d_2+1}{d_1+1}\right).$$

In order to compute $\widetilde{\sigma}_\Omega$, we set, for any $\eta\in(0,1)$,

$$\begin{aligned}A_\eta &:= \{(z,w)\in\mathbb{B}^{d_1}\times\mathbb{B}^{d_2} : ||z|| > \eta\}; \\ |A|_{\eta,M}(w) &:= \{|z|\in\mathbb{R} : (z,w)\in\Omega_M\cap A_\eta\}.\end{aligned}$$

Now, for a fixed $\eta\in(0,1)$, $\mathrm{vol}(\Omega_M\setminus A_\eta)\sim M^{-1/(d_2+1)}$ as $M\to\infty$. Therefore, for any $f\in\mathcal{C}(\overline{\Omega})$ and $\eta\in(0,1)$,

$$\lim_{M\to\infty} d_\Omega(M)\int_{\Omega_M} f\ \omega_{\mathbb{C}^d} = \lim_{M\to\infty} d_\Omega(M)\int_{\Omega_M\cap A_\eta} f\ \omega_{\mathbb{C}^d}.$$

In particular, $\lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M) = \lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M\cap A_\eta)$. Now, for any fixed $\varepsilon > 0$, we may choose η close enough to 1, so that

$$|f(r\theta, w) - f(\theta, w)| < \varepsilon \quad \text{for any } r\in(\eta,1),\ \theta\in\partial\mathbb{B}^{d_1} \text{ and } w\in\mathbb{B}^{d_2}.$$

Hence, for a fixed η and large M,

$$\begin{aligned}\int_{\Omega_M\cap A_\eta} f\ \omega_{\mathbb{C}^d} &= \int_{\mathbb{B}^{d_2}}\int_{|A|_{\eta,M}(w)}\int_{\partial\mathbb{B}^{d_1}} f(r\theta,w)s_{\mathbb{B}^d}(\theta)r^{2d_1-1}dr\ \omega_{\mathbb{C}^d}(w) \\ &< \int_{\mathbb{B}^{d_2}}\left(\int_{\partial\mathbb{B}^{d_1}}(\varepsilon + f(\theta,w))s_{\mathbb{B}^d}(\theta)\right)\int_{|A|_{\eta,M}(w)} r^{2d_1-1}dr\ \omega_{\mathbb{C}^d}(w).\end{aligned}$$

We will need the fact that for $w \in \mathbb{B}^{d_2}$,

$$\int_{|A|_{\eta,M}(w)} r^{2d_1-1}dr = \begin{cases} \frac{1}{2d_1}\left(1-\left(1-\left(\frac{K_{d_2}(w)}{\mathfrak{b}_{d_1}M}\right)^{\frac{1}{d_1+1}}\right)^{d_1}\right), & w \in B_1; \\ \frac{1}{2d_1}(1-\eta^{2d_1}), & w \in \mathbb{B}^{d_2} \setminus B_1, \end{cases}$$

where $B_1 := \{w \in \mathbb{B}^{d_2} : ||w||^2 \le 1 - (\mathfrak{b}_{d_1}\mathfrak{b}_{d_2}M(1-\eta^2)^{d_1+1})^{\frac{-1}{d_2+1}}\}$. Thus, for any fixed function h continuous in w,

$$\begin{aligned} &\int_{\mathbb{B}^{d_2}} h(w) \int_{|A|_{\eta,M}(w)} r^{2d_1-1}dr \ \omega_{\mathbb{C}^d}(w) \\ =\ & \int_{B_1} h(w) \int_{|A|_{\eta,M}(w)} r^{2d_1-1}dr \ \omega_{\mathbb{C}^d}(w) + \int_{\mathbb{B}^{d_2}\setminus B_1} h(w) \int_{|A|_{\eta,M}(w)} r^{2d_1-1}dr \ \omega_{\mathbb{C}^d}(w) \\ =\ & \frac{1}{2}(\mathfrak{b}_{d_1}M)^{\frac{-1}{d_1+1}} \int_{B_2} h(w)K_{d_2}(w)^{\frac{1}{d_1+1}} \ \omega_{\mathbb{C}^d}(w) + o(M^{\frac{-1}{d_1+1}}) + O(M^{-1/(d_2+1)}) \end{aligned}$$

as $M \to \infty$. Hence,

$$\begin{aligned} &\lim_{M\to\infty}\left(M^{\frac{1}{d_1+1}} \int_{\Omega_M \cap A_\eta} f \ \omega_{\mathbb{C}^d}\right) \\ &\qquad < \frac{1}{2}(\mathfrak{b}_{d_1}M)^{\frac{-1}{d_1+1}} \int_{\mathbb{B}^{d_2}} \int_{\partial\mathbb{B}^{d_1}} (\varepsilon + f(\theta,w))K_{d_2}(w)^{\frac{1}{d_1+1}} s_{\mathbb{B}^d}(\theta) \ \omega_{\mathbb{C}^d}(w). \end{aligned}$$

Similarly,

$$\begin{aligned} &\lim_{M\to\infty}\left(M^{\frac{1}{d_1+1}} \int_{\Omega_M \cap A_\eta} f \ \omega_{\mathbb{C}^d}\right) \\ &\qquad > \frac{1}{2}(\mathfrak{b}_{d_1}M)^{\frac{-1}{d_1+1}} \int_{\mathbb{B}^{d_2}} \int_{\partial\mathbb{B}^{d_1}} (f(\theta,w) - \varepsilon)K_{d_2}(w)^{\frac{1}{d_1+1}} \ s_{\mathbb{B}^d}(\theta) \ \omega_{\mathbb{C}^d}(w). \end{aligned}$$

We now note that $K_{d_2}^{\frac{1}{d_1+1}} \approx_{\text{const.}} (1-||w||^2)^{-\frac{d_2+1}{d_1+1}}$ is integrable on $\mathbb{B}^{d_2}$. Thus, we can let $\eta \to 1$, to obtain that for $d_\Omega(M) = M^{-1/(d_1+1)}$,

$$\widetilde{\sigma}_\Omega \ (\text{w.r.t } d_\Omega) = \frac{1}{2}(\mathfrak{b}_{d_1})^{\frac{-1}{d_1+1}} K_{d_2}^{\frac{1}{d_1+1}} \ s_{\mathbb{B}^d} \ \omega_{\mathbb{C}^d} \ (\text{as measures}).$$

□

REMARKS 4.3. An extension of the above computations shows that if $\Omega = \mathbb{B}^{d_1} \times \cdots \times \mathbb{B}^{d_k}$, where $d_1 = \cdots = d_r > d_{r+1} \ge d_{r+2} \ge \cdots d_k$, then $\widetilde{\sigma}_\Omega$ is supported on $(\partial\mathbb{B}^d)^r \times \mathbb{B}^{d_{r+1}} \cdots \times \mathbb{B}^{d_k}$ and $\widetilde{\sigma}_\Omega \approx h_{r+1} \cdots h_k \cdot (s_{\mathbb{B}^{d_1}})^r \cdot \omega_{\mathbb{C}^{d_{r+1}}} \cdots \omega_{\mathbb{C}^{d_k}}$, where $h_j(z_1, ..., z_k) = K_{d_j}^{\frac{1}{d_1+1}}(z_j)$.

We now present the proof of Theorem 1.8. We isolate a lemma that indicates how conditions (i) and (ii) help us avoid domains whose HF-gauge functions have (long-term) oscillatory behavior.

LEMMA 4.4. *Let $\Omega \Subset \mathbb{C}^d$ be such that $\alpha := \dim_{\text{HF}}(\Omega) \in (0,\infty)$ and condition (ii) of Theorem* 1.8 *holds. Then, for any $a > 0$,*

$$\lim_{M\to\infty} \frac{\text{vol}(\Omega_M)}{\text{vol}(\Omega_{Ma})} = a^{\frac{1}{\alpha}}.$$

PROOF. Set $h(M) := M^{1/\alpha}\operatorname{vol}(\Omega_M)$. Note that

$$\ell_a := \lim_{M\to\infty}\frac{h(M)}{h(aM)} = \lim_{M\to\infty}\frac{M^{\frac{1}{\alpha}}\operatorname{vol}(\Omega_M)}{(aM)^{\frac{1}{\alpha}}\operatorname{vol}(\Omega_{aM})} = a^{-\frac{1}{\alpha}}\lim_{M\to\infty}\frac{\operatorname{vol}(\Omega_M)}{\operatorname{vol}(\Omega_{Ma})}.$$

Thus, $\ell_a \in [0,\infty]$, by condition (ii).

Now, by the definition of $\dim_{\mathrm{HF}}$ and d_Ω, we know that for any $\varepsilon > 0$,

$$\begin{aligned}\lim_{M\to\infty} M^{\frac{1}{\alpha-\varepsilon}}\operatorname{vol}(\Omega_M) &= \infty\\ \lim_{M\to\infty} M^{\frac{1}{\alpha+\varepsilon}}\operatorname{vol}(\Omega_M) &= 0.\end{aligned}$$

Therefore,

$$\lim_{M\to\infty} M^{\frac{\varepsilon}{(\alpha-\varepsilon)\alpha}}h(M) = \infty \text{ and } \lim_{M\to\infty} M^{\frac{-\varepsilon}{(\alpha+\varepsilon)\alpha}}h(M) = 0. \tag{4.6}$$

Fix an $a > 1$. Suppose $\ell_a > 1$. Then, there is an $s > 0$ and an $M > 0$, such that $h(M') > a^s h(aM')$ for all $M' \geq M$. Therefore, the sequence $\{s_j := (a^jM)^s h(a^jM)\}_{j\in\mathbb{N}_+}$ is a strictly decreasing sequence of positive numbers that converges to ∞ (see the first part of (4.6)). This is a contradiction.

If $\ell_a < 1$, then, once again, for some $s > 0$ and $M > 0$, $h(M') < a^{-s}h(aM')$ for all $M' \geq M$. Therefore, the sequence $\{t_j := (a^jM)^{-s}h(a^jM)\}_{j\in\mathbb{N}_+}$ is a strictly increasing sequence of positive numbers that converges to 0 (the second part of (4.6) is invoked here). This, too, is a contradiction. Therefore, $\ell_a = 1$ when $a > 1$. When $a < 1$, we simply note that $\ell_a = 1/\ell_{\frac{1}{a}} = 1$, since $1/a > 1$. □

PROOF OF THEOREM 1.8. Fix $d_j := d_{\Omega^j}$ — a choice of HF-gauge function for Ω_j, $j = 1, 2$. We first show that $\lim_{M\to\infty} d_1(M)/d_2(M)$ exists and lies in $(0,\infty)$. For this, observe that by the condition on F, we can find $a, b > 0$ such that $a \leq |\det \mathrm{J}_{\mathbb{C}}F| \leq b$. Thus, by Proposition 3.1, $\dim_{\mathrm{HF}}(\Omega^2) = \alpha$. We set $h_j(M) := M^{1/\alpha}\operatorname{vol}(\Omega^j_M)$. Then,

$$\frac{d_1(M)}{d_2(M)} = \frac{d_1(M)\operatorname{vol}(\Omega^1_M)}{d_2(M)\operatorname{vol}(\Omega^2_M)} \times \frac{\operatorname{vol}(\Omega^2_M)}{\operatorname{vol}(\Omega^1_M)}. \tag{4.7}$$

By definition, $\lim_{M\to\infty} d_j(M)\operatorname{vol}(\Omega^j_M) \in (0,\infty)$. So, it suffices to show that $\lim_{M\to\infty}\frac{\operatorname{vol}(\Omega^2_M)}{\operatorname{vol}(\Omega^1_M)}$ is non-zero and finite (see condition (iii) for existence). Now, from the proof of Proposition 3.1 (see (3.2), in particular) we get

$$a^2\operatorname{vol}(\Omega^1_{Mb^2}) \leq \operatorname{vol}(\Omega^2_M) \leq b^2\operatorname{vol}(\Omega^1_{Ma^2}), \quad M \in (0,\infty).$$

Thus,

$$a^2\frac{\operatorname{vol}(\Omega^1_{Mb^2})}{\operatorname{vol}(\Omega^1_M)} \leq \frac{\operatorname{vol}(\Omega^2_M)}{\operatorname{vol}(\Omega^1_M)} \leq b^2\frac{\operatorname{vol}(\Omega^1_{Ma^2})}{\operatorname{vol}(\Omega^1_M)}, \quad M \in (0,\infty). \tag{4.8}$$

Thus, by Lemma 4.4, we have that $\operatorname{vol}(\Omega^2_M)/\operatorname{vol}(\Omega^1_M)$ is bounded above and below as $M \to \infty$. Combining (4.7), (4.8) and (iii),

$$L := \lim_{M\to\infty}\frac{d_2(M)}{d_1(M)} \text{ exists and is in } (0,\infty). \tag{4.9}$$

Now, in order to prove the transformation law, we first show that the measure $F^*\widetilde{\sigma}_{\Omega^2}$ is absolutely continuous with respect to $\widetilde{\sigma}_{\Omega^1}$. For this, we set

$$\sigma^j_M := d_j(M)\chi_{\Omega^j_M}\,\omega_{\mathbb{C}^d}, \qquad j = 1, 2.$$

We also recall that if a bounded family of positive Borel measures $\{\ell_M\}_{M>0}$ on a metric space X converges weakly to a finite positive measure σ on X, then

(4.10)

$$\lim_{M\to\infty} \ell_M(C) = \sigma(C) \text{ for every continuity set } C \text{ — i.e., } \sigma(\partial C) = 0 \text{ — of } X.$$

Now, let $A \subset \overline{\Omega^1}$ be such that $\widetilde{\sigma}_{\Omega^1}(A) = 0$, and $\varepsilon > 0$. By the sparseness of discontinuity sets (see [**16**, Page 7]) and the regularity of $\widetilde{\sigma}_{\Omega^1}$, we can find open sets V_ε in $\overline{\Omega^1}$ containing A such that $\widetilde{\sigma}_{\Omega^1}(V_\varepsilon) < \varepsilon$, and V_ε are continuity sets for $\widetilde{\sigma}_{\Omega^1}$ and $F^*\widetilde{\sigma}_{\Omega^2}$. By (4.10),

$$\lim_{M\to\infty} \ell^1_M(V_\varepsilon) = \widetilde{\sigma}_{\Omega^1}(V_\varepsilon) < \varepsilon.$$

By (3.1) in the proof of Proposition 3.1, we observe that $F^{-1}(F(V_\varepsilon) \cap \Omega^2_M) \subset V_\varepsilon \cap \Omega^1_{Ma^2}$. Hence,

$$\begin{aligned} F^*\sigma^2_M(V_\varepsilon) &\leq b^2 \frac{d_2(M)}{d_1(Ma^2)} \sigma^1_{Ma^2}(V_\varepsilon) \\ &= b^2 \frac{d_2(M)}{d_1(M)} \frac{d_1(M)\operatorname{vol}(\Omega^1_M)}{d_1(Ma^2)\operatorname{vol}(\Omega^1_{Ma^2})} \frac{\operatorname{vol}(\Omega^1_{Ma^2})}{\operatorname{vol}(\Omega^1_M)} \sigma^1_{Ma^2}(V_\varepsilon). \end{aligned}$$

As $d_2(M)/d_1(M)$, $d_1(M)\operatorname{vol}(\Omega^1_M)$ and $\operatorname{vol}(\Omega^2_{Ma^2})/\operatorname{vol}(\Omega^1_M)$ all admit finite, non-zero limits as $M \to \infty$, we get that $F^*\sigma^2_M(V_\varepsilon) < c\varepsilon$ for large enough M, and some constant $c > 0$ independent of ε and M. By (4.10), $F^*\widetilde{\sigma}_{\Omega^2}(V_\varepsilon) = \lim_{m\to\infty} F^*\sigma^2_M(V_\varepsilon) < c\varepsilon$. By outer regularity, $F^*\widetilde{\sigma}_{\Omega^2}(A) = 0$.

In view of the Radon-Nikodym theorem, our conclusion above shows that there exists a $\widetilde{\sigma}_{\Omega^1}$-measurable function G on $\partial\Omega^1$ such that $F^*(\widetilde{\sigma}_{\Omega^2}) = G \cdot \widetilde{\sigma}_{\Omega^1}$ on $\partial\Omega^1$. Let $x_0 \in \partial\Omega^1$. By the sparseness of discontinuity sets, we may find a decreasing sequence of neighborhoods U_ε of x_0 that are continuity sets with respect to both $\widetilde{\sigma}_{\Omega^1}$ and $F^*\widetilde{\sigma}_{\Omega^2}$ and satisfy

$$|\det \mathrm{J}_{\mathbb{C}}F(x) - \det \mathrm{J}_{\mathbb{C}}F(x_0)| < \varepsilon \ \forall x \in U_\varepsilon.$$

Now, we observe that

$$\begin{aligned} F^{-1}(\Omega^2_M \cap F(U_\varepsilon)) &= \{z \in \Omega^1 \cap U_\varepsilon : K_2(F(z)) > M\} \\ &= \{z \in \Omega^1 \cap U_\varepsilon : K_1(z) > M|\det \mathrm{J}_{\mathbb{C}}F(z)|^2\} \\ &\subseteq \{z \in \Omega^1 \cap U_\varepsilon : K_1(z) > M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)| - \varepsilon)^2\}. \end{aligned}$$

As in (3.2), we get that

(4.11)

$$F^*\sigma^2_M(U_\varepsilon) \leq (|\det \mathrm{J}_{\mathbb{C}}F(x_0)| + \varepsilon)^2 \frac{d_2(M)}{d_1(M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)| - \varepsilon)^2)} \sigma^1_{M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon)^2}(U_\varepsilon).$$

In a similar manner, we get

(4.12)

$$F^*\sigma^2_M(U_\varepsilon) \geq (|\det \mathrm{J}_{\mathbb{C}}F(x_0)| - \varepsilon)^2 \frac{d_2(M)}{d_1(M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)| + \varepsilon)^2)} \sigma^1_{M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon)^2}(U_\varepsilon).$$

Taking limits as $M \to \infty$ on both sides of (4.11) and (4.12), observing that

$$\lim_{M\to\infty} \frac{d_2(M)}{d_1(cM)} = \lim_{M\to\infty} \left(\frac{d_2(M)}{d_1(M)} \frac{d_1(M)\operatorname{vol}(\Omega^1_M)}{d_1(cM)\operatorname{vol}(\Omega^1_{cM})} \frac{\operatorname{vol}(\Omega^1_{cM})}{\operatorname{vol}(\Omega^1_M)} \right) = c^{-1/\alpha} L$$

due to (4.9), the defining property of d_1, and Lemma 4.4, we get that

$$L\left(\frac{|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon}{(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon)^{-1/\alpha}}\right)^2 \le \frac{F^*\widetilde{\sigma}_{\Omega^2}(U_\varepsilon)}{\widetilde{\sigma}_{\Omega_1}(U_\varepsilon)} \le L\left(\frac{|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon}{(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon)^{-1/\alpha}}\right)^2.$$

Therefore, as $\varepsilon \to 0$, we get that

$$F^*\widetilde{\sigma}_{\Omega^2}\ (\text{w.r.t. } d_2) = L|\det \mathrm{J}_{\mathbb{C}}F|^{2\left(1-\frac{1}{\alpha}\right)}\widetilde{\sigma}_{\Omega_1}(\text{w.r.t. } d_1) \qquad (\text{a.e. w.r.t. } \widetilde{\sigma}_{\Omega^1}),$$

where $L = \lim_{M\to\infty} d_2(M)/d_1(M)$. □

Acknowledgements. This work forms a part of the author's doctoral thesis. She is grateful to her thesis advisor, David Barrett, for persuading her to explore this avenue, and for his insightful comments on this article. She would also like to thank the anonymous referee for his/her suggestions which vastly improved the presentation of this article.

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