MA221 HOMEWORK ASSIGNMENT 2

Due Date: August 30 (Mon.) by 11:59 pm

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1. **Only the four problems marked with a $*$ will be graded.**
2. Your assignment may be hand-written or typed, but it must be submitted via MS Teams in the form of a **single PDF document**.
3. Unless otherwise stated, you may only use definitions/theorems/facts stated in class.

Problem A. In class, we did not prescribe how to multiply two Dedekind cuts. We will (partially) tackle this here. Given positive cuts A and B, define

$$A \cdot B = \{q \in \mathbb{Q} : q \leq rs \text{ for some choice of } r \in A, s \in B,\ r, s > 0\}.$$

(a) Show that $A \cdot B$ is a positive cut.

(b) Check that the operation $\cdot$ is commutative and associative for all positive cuts.

(c) Show that the cut $\{q \in \mathbb{Q} : q < 1\}$ acts as the multipicative identity in this setting.

(d) Given A, describe the multiplicative inverse of A. Please provide a complete justification for your answer.

Problem B. Show that $(\mathbb{Q}, \leq)$ is the smallest ordered field in the following sense: given any ordered field $(X, \leq)$, $\mathbb{Q}$ admits a field isomorphism onto a subfield of X, where the isomorphism also preserves the order.

Problem C$*$. Consider the following relation on $\mathbb{C} \times \mathbb{C}$:

$$(a + ib) \leq (c + id) \quad \iff \quad (a < c) \text{ or } (a = c \text{ and } b \leq d).$$

(a) Show that $\leq$ is a (total) order on $\mathbb{C}$.

(b) Does $(\mathbb{C}, \leq)$ have the least upper bound property? Justify your answer.

Problem D. Let $x \in (0, 1)$.

(a) Show that x has a nonunique decimal representation if and only if it can be written as $\frac{p}{10^q}$ for some integers p, q. If x is indeed of this form, show that there exist exactly two decimal representations of x.

($b*$) Show that x is rational if and only if the digits in its decimal representation,

$$x = 0.d_1 d_2 d_3 ...,$$

start repeating, i.e., there exist $N, r \in \mathbb{N}_{>0}$ such that $d_n = d_{n+r}$ for all $n \geq N$. (*Note. You may use Part* (a) *without proof when submitting your solution to Part* (b).)

Problem E. For a vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, $||\mathbf{x}|| := (x_1^2 + \cdots + x_n^2)^{1/2}$. Show that, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the following hold.

(a) $\Big| ||\mathbf{x}|| - ||\mathbf{y}|| \Big| \leq ||\mathbf{x} - \mathbf{y}||$.

(b) $||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 = 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2$.

Problem F∗. Show that the set of all finite subsets of a countable set is a countable set.

Problem G∗. Let $E = \{0, 1, 2\}$, $n \in \mathbb{N}$ and $S = E^n$. Let $\chi : E \times E \to \{0, 1\}$ be the function

$$\chi(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

Given $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$, let

$$d(x, y) = \sum_{j=1}^{n} \chi(x_j, y_j).$$

Is (S, d) a metric space? As always, you must completely justify your answer.