MA221 HOMEWORK ASSIGNMENT 3

Due date: September 17 (Fri.) by 11:59 pm

---

1. **Only the five problems marked with a $*$ will be graded.**
2. Your assignment may be hand-written or typed, but it must be submitted via MS Teams in the form of a **single PDF document**.
3. Unless otherwise stated, you may only use definitions/theorems/facts stated in class.

Problem A. Consider the following claims for a metric space (X, d). For each claim, determine whether it is true or false. You must justify your answer in either case.

(a) Suppose $S \subseteq T \subseteq X$. Then, S is compact in the metric space (X, d) if and only if S is compact in $(T, d|_T)$, where $d|_T$ is the subspace metric on T.

($b*$) The set of limit points of a set is always closed.

($c*$) The interior of a connected set is always connected.

Problem B. In class, we showed that limit-point compactness implies compactness for subsets of $\mathbb{R}^n$. Complete the following steps to establish this result for general metric spaces.

Let (X, d) be a metric space. Suppose every infinite subset of X has a limit point in X.

(a) Show that X has a countable dense subset, i.e., there is a countable set $S \subseteq X$ such that $\overline{S} = X$.

(b) Show that there is a countable collection $\{U_n\}_{\alpha \in \mathbb{N}}$ of open sets such that for any open subset $U \subseteq X$ and any $x \in U$, there is an $n \in \mathbb{N}$ such that $x \in U_n \subseteq U$.

($c*$) Now, let $\mathscr{V} = \{V_\alpha\}_{\alpha \in \Lambda}$ be an arbitrary open cover of X. Using Parts (a) and (b), argue that $\mathscr{V}$ admits a countable subcover of X. Now, argue that this countable subcover admits a finite subcover of X.

You do not need to submit the proofs of (a) *and* (b). *You may invoke them directly in your proof for* (c).

Problem C. In each case below, produce an example (with justifications).

($a*$) A metric space (X, d) where $\overline{B(a; r)} \neq \{x \in X : d(x, a) \leq r\}$ for some $a \in X$, $r > 0$.

(b) A metric space (X, d) with a closed and bounded, but non-compact set $E \subseteq X$.

(c) A metric space (X, d) that does not contain a countable dense subset.

Problem D$*$. Let $C \subset [0, 1]$ be the Cantor set (as constructed in class). Show that $C + C = [0, 2]$.

Problem E. Let $r > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $x_0 = r$ and

$$x_{n+1} = x_n - \frac{x_n^2 - r}{2x_n}, \quad n \geq 1. \tag{1}$$

Show that $\lim_{n\to\infty} x_n = \sqrt{r}$. *Note. As part of this proof, you must also confirm that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is well-defined, i.e., you are not dividing by 0 in (1).*

Problem F*. Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$, with $x_n \geq 0$, for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}_{>0}$. Show that

$$\lim_{n\to\infty} (x_n)^{\frac{1}{k}} = \left(\lim_{n\to\infty} x_n\right)^{\frac{1}{k}}.$$