MA221 PRACTICE PROBLEMS FOR TEST II

Problem A. Let $X = \{\{x_n\}_{n\in\mathbb{N}} : x_n \in \{0,1\} \text{ for all } n \in \mathbb{N}\}$. Define $d : X \times X \to [0,\infty)$ as follows

$$d(\{x_n\},\{y_n\}) = \max_{n\in\mathbb{N}} |x_n - y_n|.$$

Check that d is a metric on X. Is X a compact metric space?

Problem B. Let (X, d) be a complete metric space. Suppose $E \subset X$ is endowed with the subspace metric $d|_E$. Show that $(E, d|_E)$ is a complete metric space if and only if E is closed in X.

Problem C. Recall that a set $E \subset \mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if $\overline{E} = \mathbb{R}$. Using the Nested Interval Property of $\mathbb{R}$ prove the following: if $\{U_n\}_{n\in\mathbb{N}}$ is a collection of open, dense subsets of $\mathbb{R}$, then

$$U = \bigcap_{n\in\mathbb{N}} U_n$$

is dense in $\mathbb{R}$.

Problem D. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence. Define

$$X_n = \sup\{x_k : k \geq n\}, \qquad n \in \mathbb{N}.$$

Show that $\{X_n\}_{n\in\mathbb{N}}$ admits a limit in the extended real line, and that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} X_n.$$

Problem E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Suppose $x \in X$ is such that every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to x. Show that, then

$$\lim_{n\to\infty} x_n = x.$$

Problem F. Let $x > 0$ and $q \in \mathbb{R}$. Suppose $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of rational numbers such that $\lim_{n\to\infty} q_n = q$. Show that, then $\{x^{q_n}\}_{n\in\mathbb{N}}$ is a convergent sequence. **Note.** This allows us to define x^q for irrational q.

Problem G. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of **positive** real numbers such that $x_n \neq 1$ for any $n \in \mathbb{N}$. Show that

$$\sum_{n=1}^{\infty} x_n < \infty \iff \sum_{n=1}^{\infty} \frac{x_n}{1+x_n} < \infty.$$