MA224 HOMEWORK ASSIGNMENT 2

Due Date: March 10 (Wed.) by 11:59 pm

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1. **Mode of submission.** Assignments must be submitted via teams (as discussed in class). Your assignment may be hand-written or typed, but the final submission must be in the form of **a PDF document**.
2. **Grading scheme.** You are expected to submit solutions to all the problems. The grader will grade selected problems.
3. **What can I use?** As much as possible, use definitions/theorems/facts stated in class. Avoid heavy machinery — you'll be alerted when it is needed.
4. **On collaborative efforts.** Teamwork is absolutely discouraged. These assignments are meant to test your individual understanding of the course material.

Problem 1. Determine (with brief justifications) which of the following identities are true.

(a) $\log(zw) = \log(z) + \log(w)$, where log denotes the principal branch of the logarithm.

(b) $|\cos(z)|^2 = \cos^2(\operatorname{Re} z) + \sin^2(i \operatorname{Im} z)$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-analytic function. We abuse notation and also denote the line $\{z \in \mathbb{C} : \operatorname{Im} z = 0\}$ by $\mathbb{R}$ below.

(a) Suppose $F : U \to \mathbb{C}$ is a complex-analytic function on some neighborhood U of $\mathbb{R}$ in $\mathbb{C}$ such that $F|_{\mathbb{R}} = f$. Show that the radius of convergence of the (complex) Taylor series of F about any $x + i0 \in \mathbb{R}$ coincides with the radius of convergence of the (real) Taylor series of f about x.

(b) For any fixed $x_0 \in \mathbb{R} \setminus \{0\}$, give lower and upper bounds (in terms of x_0) on the radius of convergence of the Taylor series of $f(x) = \dfrac{1}{1+x^2}$ about x_0.

Remark. You may assume the fact that $1/(1+z^2)$ is complex-analytic on $\mathbb{C} \setminus \{\pm i\}$.

Problem 3. A twice-differentiable function $u : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ is said to be *harmonic* on U if

$$\triangle u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0 \quad \text{on } U.$$

Convince yourself that the real and imaginary parts of a twice-differentiable holomorphic function on U are harmonic. Use this fact to determine which of the following choices of u cannot be the real part of a holomorphic function (on any domain). For the other function, produce an explicit holomorphic function F such that $\operatorname{Re} F = u$ and $F(0) = 0$.

(a) $u(x, y) = \dfrac{x^2 + y^2}{1 + 2x + x^2 + y^2}$.

(b) $u(x, y) = \dfrac{x(1 + x^2 + y^2)}{1 + 2x^2 - 2y^2 + (x^2 + y^2)^2}$.

(Hint. You may find it convenient to express everything (including $\triangle$) in terms of z and $\overline{z}$.)

Problem 4. Show that if a function $f : D(0; R) \to \mathbb{C}$ is Fréchet-differentiable on $D(0; R)$ for some $R > 0$, then

$$\lim_{r \to 0} \frac{1}{r^2} \int_{\{|z|=r\}} f(z)\, dz = 2\pi i \frac{\partial f}{\partial \overline{z}}(0).$$