MA224 HOMEWORK ASSIGNMENT 3

Due Date: March 21 (Sun.) by 11:59 pm

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1. **Mode of submission.** Assignments must be submitted via teams (as discussed in class). Your assignment may be hand-written or typed, but the final submission must be in the form of **a PDF document**.
2. **Grading scheme.** You are expected to submit solutions to all the problems. The grader will grade selected problems.
3. **What can I use?** As much as possible, use definitions/theorems/facts stated in class. Avoid heavy machinery — you'll be alerted when it is needed.
4. **On collaborative efforts.** Teamwork is absolutely discouraged. These assignments are meant to test your individual understanding of the course material.

Problem 1. Compute the following integrals (either by direct computation, or by applying any of the results stated in class). The curves are to be given anti-clockwise orientation (and should be traversed once).

(1) $\displaystyle\int_{C(0;1)} \frac{1}{|\zeta - a|^2}\,|d\zeta|$, where $|a| \neq 1$. (Hint. Express $|d\zeta|$ in terms of $d\zeta$ for $|\zeta| = 1$.)

(2) $\displaystyle\int_{C(0;2)} \frac{\zeta}{(\zeta - 1)(\zeta - 1 + 2i)}\,d\zeta$.

Problem 2. Suppose R is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_0 \neq 0$. Show that for any fixed $0 < r < R$, the function f does not vanish on the disk $D(0; L_r)$, where

$$L_r = \frac{|a_0| r}{|a_0| + M} \qquad \text{and} \qquad M = \sup_{z \in C(0;r)} |f(z)|.$$

Problem 3. Suppose f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$. Using the Cauchy integral formula, show that

$$f(z) = \frac{1}{2\pi} \int_{C(0;1)} f(\zeta) \left(\frac{1 - |z|^2}{|\zeta - z|^2} \right) \frac{d\zeta}{i\zeta},$$

where $C(0;1)$ is given the standard orientation. Next, using the formula above, show that if f is $\mathbb{R}$-valued on $b\mathbb{D} = C(0;1)$, then f must be $\mathbb{R}$-valued in $\mathbb{D}$.

(Hint. What can you say about $\displaystyle\int_{C(0;1)} \frac{f(\zeta)}{\zeta - (1/\overline{z})}\,d\zeta$ when $z \in \mathbb{D}$?)

Problem 4. Let $z_1, z_2, ..., z_n \in \mathbb{C}$ be distinct points in a disc $D(a;r)$ and f be a holomorphic in a disc containing $\overline{D(a;r)}$. Then the unique polynomial $P(z)$ of degree $n-1$ which satisfies $P(z_j) = f(z_j)$ for all $j = 1, ..., n$ is given by

$$P(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{q(\zeta)} \frac{q(\zeta) - q(z)}{\zeta - z} \, d\zeta \tag{1}$$

for a suitably chosen polynomial q. Find q, and show that P defined as in equation (1) does indeed satisfy $P(z_j) = f(z_j)$ for all $j = 1, ..., n$.

Remark. The above claim would be complete if you also showed that P is a degree $n-1$ polynomial. Think about why this is the case (but don't submit an answer to this part).