MA224 HOMEWORK ASSIGNMENT 4

Due Date: April 02 (Fri.) by 11:59 pm

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1. **Mode of submission.** Assignments must be submitted via teams (as discussed in class). Your assignment may be hand-written or typed, but the final submission must be in the form of **a PDF document**.
2. **Grading scheme.** You are expected to submit solutions to all the problems. The grader will grade selected problems.
3. **What can I use?** As much as possible, use definitions/theorems/facts stated in class. Avoid heavy machinery — you'll be alerted when it is needed.
4. **On collaborative efforts.** Teamwork is absolutely discouraged. These assignments are meant to test your individual understanding of the course material.

Problem 1. Let $v \in \mathbb{C}$ and $w \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Let $\Lambda = \{mv + nw : m, n \in \mathbb{Z}\}$. Suppose $f \in \mathcal{O}(\mathbb{C})$ is such that $f(z + \lambda) = f(z)$ for every $z \in \mathbb{C}$ and $\lambda \in \Lambda$. Show that f is a constant function on $\mathbb{C}$.

Problem 2. Let $\Omega \subset \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$ be a nonvanishing function such that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz \equiv 0 \mod 2$$

for all closed curves $\gamma \subset \Omega$.

(a) Does the above condition guarantee the existence of a holomorphic logarithm of f, i.e., a $g \in \mathcal{O}(\Omega)$ such that $e^g = f$ on Ω?

(b) Show that the above condition guarantees the existence of a holomorphic square root of f, i.e., there is an $h \in \mathcal{O}(\Omega)$ such that $h^2 = f$.

Hint. For (b), try to construct an h of the form e^p, where p involves an integral of f'/f.

Problem 3. Given $f \in \mathcal{C}(\Omega)$, let

$$(f)_r := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta \right)^{\frac{1}{2}}.$$

Let

$$H^2(\mathbb{D}) = \left\{ f \in \mathcal{O}(\mathbb{D}) : ||f||_{H^2} := \sup_{0<r<1} (f)_r < \infty \right\}.$$

(a) Characterize the functions in $H^2(\mathbb{D})$ in terms of the coefficients of their power series expansions around 0, i.e., when does $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belong to $H^2(\mathbb{D})$? Further, write $||f||_{H^2}$ in terms of $\{a_n\}_{n\geq 0}$.

(b) Show that, for any compact set $K \subset \mathbb{D}$, there is a constant $C_K > 0$ such that

$$\sup_{z\in K} |f(z)| \leq C_K ||f||_{H^2}, \qquad \forall f \in H^2(\mathbb{D}).$$

(c) Assuming $d(f,g) := ||f-g||_{H^2}$ is a metric on $H^2(\mathbb{D})$, show that if $\{f_n\}_{n\in\mathbb{N}} \subset H^2(\mathbb{D})$ is Cauchy with respect to the metric d, then there is an $f \in \mathcal{O}(\mathbb{D})$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on compacts to f on $\mathbb{D}$.

(d) In (c) above, must f belong to $H^2(\mathbb{D})$?