# MA224 HOMEWORK ASSIGNMENT 5

**Due Date: April 11 (Sun.) by 11:59 pm**

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1. **Mode of submission.** Assignments must be submitted via teams (as discussed in class). Your assignment may be hand-written or typed, but the final submission must be in the form of **a PDF document**.
2. **Grading scheme.** You are expected to submit solutions to all the problems. The grader will grade selected problems.
3. **What can I use?** As much as possible, use definitions/theorems/facts stated in class. Avoid heavy machinery — you'll be alerted when it is needed.
4. **On collaborative efforts.** Teamwork is absolutely discouraged. These assignments are meant to test your individual understanding of the course material.

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**Problem 1.** Let $\Omega \subset \mathbb{C}$ be an open set. Suppose $\gamma_0$, $\gamma_1 : [0,1] \to \Omega$ are curves with the same endpoints. Let $\gamma : [0,1] \to \Omega$ be the curve

$$\gamma(t) = \begin{cases} \gamma_0(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma_1(2-2t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that $\gamma_0$ and $\gamma_1$ are f.e.p. homotopic in $\Omega$ if and only if $\gamma$ is nullhomotopic in $\Omega$.

**Problem 2.** Recall that the stereographic projection

$$\Psi : (x,y,t) \mapsto \begin{cases} \frac{x+iy}{1-t}, & \text{when } (x,y,t) \neq (0,0,1), \\ \infty, & \text{when } (x,y,t) = (0,0,1), \end{cases}$$

identifies the unit sphere $S^2$ in $\mathbb{R}^3$ with the extended complex plane $\hat{\mathbb{C}}$. Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2;\mathbb{C})$, the fractional linear transformation $M_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as

$$M_A : z \mapsto \begin{cases} \dfrac{az+b}{cz+d}, & \text{if } z \neq -\frac{d}{c}, \infty, \\ \infty, & \text{if } z = -\frac{d}{c}, \\ \dfrac{a}{c}, & \text{if } z = \infty, \end{cases} \qquad (c \neq 0),$$

and as $M_A = (az+b)/d$ with $M_A(\infty) = \infty$, when $c = 0$.

(*a*) The spherical distance on $\hat{\mathbb{C}}$ is given by $d_s(z,w) := ||\Psi^{-1}(z) - \Psi^{-1}(w)||_{\mathbb{R}^3}$. Compute an expression for $d_s(z,w)$ (including an expression for $d_s(z,\infty)$, $z \in \mathbb{C}$).

(*b*) Now show that if $\det A = 1$, then $M_A$ preserves the spherical distance on $\hat{\mathbb{C}}$ if and only if $d = \overline{a}$ and $c = -\overline{b}$.

**Problem 3.** Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function satisfying an estimate

$$|f(z)| \leq Ce^{\alpha|\operatorname{Im} z|},$$

for some $C > 0$ and $\alpha \in (0, \pi)$. Show that

$$f(z) = \lim_{n\to\infty} \sum_{n=-N}^{n=N} f(n) \frac{\sin(\pi(z-n))}{\pi(z-n)}$$

for all $z \in \mathbb{C}$. Here, one interprets $\frac{\sin(z)}{z}|_{\{z=0\}}$ as 1.

*For any noninteger $z \in \mathbb{C}$, apply the residue theorem to the function $g(w) = \frac{f(w)}{(w-z)(\sin(\pi w))}$ on the boundaries of appropriate rectangles.*