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CONVEX FLOATING BODIES AS APPROXIMATIONS OF BERGMAN SUBLEVEL SETS ON TUBE DOMAINS

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ABSTRACT. For a pseudoconvex tube domain, we prove estimates that relate the sublevel sets of its diagonal Bergman kernel to the floating bodies of its convex base. This allows us to associate a new affine invariant to any convex body.

1. INTRODUCTION

The main objective of this paper is to establish a quantitative relationship between two collections of geometric objects associated with a given convex body in $\mathbb{R}^n$. One, its set of convex floating bodies — an equiaffine-invariant construction studied by convex geometers, and the other, the collection of sublevel sets traced by the Bergman kernel of a tube domain over the given body. The latter is a natural object in complex analysis. Although, the bridge between convex and complex analysis on such domains has been exploited succesfully before — Nazarov's paper [8] is a noteworthy example — the role of floating bodies in this interplay is yet to be explored. Before we state our main result, we describe the central objects of this paper in some detail.

Let $D \subset \mathbb{R}^n$ be a bounded convex domain. For $\delta > 0$, its *convex floating body* D_δ is the intersection of all the half-spaces whose defining hyperplanes cut off a set of volume δ from D. Specifically, if A denotes the set of all $(v,t) \in \mathbb{R}^n \times \mathbb{R}$ such that $\mathrm{vol}\{x \in D : x \cdot v \geq t\} = \delta$, then

$$D_\delta = \bigcap_{(v,t)\in A} \{x \in \mathbb{R}^n : x \cdot v < t\}. \tag{1.1}$$

These are strictly convex and exhaust D as δ approaches zero. Inspired by a construction due to Dupin, these were first introduced by Schütt and Werner (in [12]) as a tool for extending the notion of Blaschke's surface area measure to nonsmooth convex boundaries. Since its introduction, the floating body has made appearances in the context of polyhedral approximations (see [11]), the homethety conjecture (see [14] and [15]) and, more recently, the hyperplane conjecture (in [3]).

Now, let $\Omega := \{x + iy \in \mathbb{C}^n : y \in D\}$. Then, Ω is a pseudoconvex tube domain in $\mathbb{C}^n$. The Bergman kernel of Ω, $K_\Omega : \Omega \times \Omega \to \mathbb{C}$, is the reproducing kernel of the Bergman space $A(\Omega)$ — i.e., the space of Lebesgue square-integrable holomorpic functions on Ω, with the L^2-norm. It is known that $A(\Omega)$ is nonempty, consists of

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Fourier-Laplace transforms of certain functions on $\mathbb{R}^n$, and

$$K_\Omega(z,w) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(z-\overline{w})\cdot t}}{\int_D e^{-2x\cdot t}d\mu(x)}d\mu(t), \tag{1.2}$$

where μ denotes the Lebesgue measure on $\mathbb{R}^n$ (see [10], and the references therein). Estimates for the Bergman kernel and associated quantities are of great interest to complex analysts and are the subject of many works. We will focus on the sets

$$D^M := \{x \in D : K_D(x) := K_\Omega(ix, ix) < M\}. \tag{1.3}$$

These are strongly convex (this follows from the strict plurisubharmonicity of $\log K_\Omega(z, z)$) and exhaust D as $M \to \infty$ (as discussed in Section 2). Although (1.2) gives a formula for $K_D(x)$, it can be hard to compute, even for some very simple examples (such as planar triangles). On the other hand, the convex floating bodies are simpler to construct and visualize. This is part of our motivation for establishing the following relation:

Theorem 1.1. *Let $D \subseteq \mathbb{R}^n$ be a bounded convex domain. Let D_δ and D^M be the δ-convex floating body and the Bergman M-sublevel set of D, respectively (see* (1.1) *and* (1.3)*). Then, there exist dimensional constants $\ell_n > 0$ and $u_n > 0$ such that*

$$D^{\ell_n\delta^{-2}} \subseteq D_\delta \subseteq D^{u_n\delta^{-2}}$$

for small enough δ.

Another reason to compare these two collections is their suitability for the following scheme. Suppose G is a group of volume-preserving transformations that acts on $\mathbb{R}^n$ (or $\mathbb{C}^n$), and $D \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) is a bounded domain. If $\{D(\varepsilon)\}_{\varepsilon>0}$ is a G-invariant collection of exhausting subsets of D, and

$$\text{vol}(D \setminus D(\varepsilon)) \sim f(\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

for some continuous f with $f(0) = 0$, then the weak-$*$ limit (if it exists) of $f(\varepsilon)^{-1}$ times the Lebesgue measure on $D \setminus D(\varepsilon)$ yields a G-invariant measure on ∂D. If D is strongly convex and $D(\varepsilon)$ is chosen as the convex floating body D_ε, then this measure is the normalized affine surface area measure on ∂D (this is implicit in Schütt and Werner's paper [12]). For other convex domains, the floating bodies can lead to 'lower-dimensional' affine measures (for instance, this measure is supported on the vertices in the case of polygons — see [11]). If the above scheme is carried out for a strongly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$, using the Bergman sublevel sets, then one obtains the normalized Fefferman hypersurface measure on $\partial\Omega$ (see [5, Prop. 1.5]). If D is strongly convex, the tube domain $\Omega := \mathbb{R}^n + iD$ is strongly pseudoconvex, and the Fefferman measure on $\partial\Omega$ reduces to the affine measure along ∂D. It follows that if $D(\varepsilon)$ is set as the Bergman sublevel set $D^{1/\varepsilon}$, then, again we obtain the normalized affine surface area measure on ∂D. Theorem 1.1 implies that, analogous to strongly convex domains, the two competing classes $\{D_\varepsilon\}$ and $\{D^{1/\varepsilon}\}$ will yield comparable equiaffine-invariant measures on ∂D for a general convex domain $D \subset \mathbb{R}^n$. This is surprising since in the absence of strong convexity, we do not have any Schütt-Werner or Hörmander-type estimates relating these sets to the curvature of ∂D (the estimates referred to are used in the proof of Proposition 3.2).

The rest of the article is organized as follows. We provide a proof of Theorem 1.1 in the next section. The constants ℓ_n and u_n are computed therein. In Section 3, we set up a new affine-invariant constant associated to a convex body, and compute it for some examples. At the end, we indicate some possible avenues of future exploration.

2. Proof of Theorem 1.1

Notation. We first clarify some notation that will appear throughout the rest of this article. We use $\mathbb{B}^n$ and ω_n to denote the unit Euclidean ball and its volume, respectively, in $\mathbb{R}^n$. The unit disc in $\mathbb{C}$ is written as $\mathbb{D}$. The space of holomorphic maps from D_1 to D_2 is denoted by $\mathcal{O}(D_1; D_2)$. For complex-valued n-tuples $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, $a \cdot b = a_1 b_1 + \cdots + a_n b_n$.

We now briefly argue the fact that the Bergman sublevel sets $\{D^M\}_{M>0}$ exhaust D. Although, this is not necessary for our main proof, it is an essential feature of the comparison we are making between $\{D_\varepsilon\}_{\varepsilon>0}$ and $\{D^M\}_{M>0}$.

Lemma 2.1. *Let $D \subset \mathbb{R}^n$ be a bounded convex domain. Then, for any $x_0 \in \partial\Omega$, $K_D(x) \to \infty$ as $x \to x_0$.*

Proof. Let $R := \{(z_1, \dots, z_n) : (\log|z_1|, \dots, \log|z_n|) \in D\}$. As D is a bounded convex domain, R is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$ that satisfies the Fu condition — i.e., it does not intersect any complex hyperplane of the form $\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = 0\}$. Thus, R is hyperconvex, and $K_R(z, z) \to \infty$ as $z \mapsto z_0$, for any $z_0 \in \partial R$ (these are results from [16] and [9], respectively). Now, by Theorem 2 and estimate (7) in Fu's paper [4],

$$K_R\big((e^{x_1}, \dots, e^{x_n}), (e^{x_1}, \dots, e^{x_n})\big) = e^{-2(x_1+\cdots+x_n)} \sum_{k \in \mathbb{Z}^n} K_{\mathbb{R}^n+iD}(ix, ix + 2k\pi)$$
$$\leq C K_{\mathbb{R}^n+iD}(ix, ix) \sum_{\substack{k \in \mathbb{Z}^n, \\ k \neq (0,\cdots,0)}} \frac{1}{|k|^2},$$

for $x = (x_1, \dots, x_n) \in D$, and some constant C independent of x. Thus, $K_D(x) \geq \widetilde{C} K_R\big((e^{x_1}, \dots, e^{x_n}), (e^{x_1}, \dots, e^{x_n})\big)$, where $\widetilde{C}$ is independent of x. Combining this with the hyperconvexity of R, we get the desired result. ☐

We now proceed to the proof of our main theorem. We rely on Nazarov's approach from [8], the main source of challenge being the lack of any symmetry assumptions on D.

Proof of Theorem 1.1. Let D be a bounded convex domain in $\mathbb{R}^n$. We first establish the existence of u_n. For this, we repeat an estimate due to Nazarov (see [8, Section 3]). Let $E \subset \mathbb{R}^n$ be an origin-symmetric convex body. One uses formula (1.2) to write

$$K_E(0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{J_E(t)} d\mu(t) \tag{2.1}$$

where

$$J_E(t) = \int_E e^{-2x\cdot t} d\mu(x).$$

Fix a $y \in E$. Then, $E^y := \frac{1}{2}(y+E) \subseteq E$. So, we obtain

$$\begin{aligned} J_E(t) \geq \int_{E^y} e^{-2x\cdot t} d\mu(x) &= 2^{-n} \int_E e^{-2(\frac{v+y}{2})\cdot t} d\mu(v) \\ &= 2^{-n} e^{-y\cdot t} \int_E e^{-v\cdot t} d\mu(v) \\ &\geq 2^{-n} e^{-y\cdot t} \operatorname{vol}(E), \end{aligned} \tag{2.2}$$

where we use the convexity of $v \mapsto e^{-v\cdot t}$ on E for every t, and the observation that any convex function f on E satisfies

$$\int_E f(x) d\mu(x) \geq f(0) \operatorname{vol}(E)$$

by the symmetry of E. Next, recall that the polar body of E is given by $E^\circ = \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in E\}$, and

$$\begin{aligned} ||x||_{E^\circ} &:= \min\{\alpha > 0 : x \in \alpha E^\circ\} \\ &= \max\{x \cdot y : y \in E\}. \end{aligned}$$

So, maximizing (2.2) over all $y \in E$, we obtain that

$$J_E(t) \geq 2^{-n} e^{||-t||_{E^\circ}} \operatorname{vol}(E) = 2^{-n} e^{||t||_{E^\circ}} \operatorname{vol}(E),$$

for all $t \in \mathbb{R}^n$. Substituting this back in (2.1), we see that

$$\begin{aligned} K_E(0) &\leq \frac{1}{\pi^n \operatorname{vol}(E)} \int_{\mathbb{R}^n} e^{-||t||_{E^\circ}} d\mu(t) \\ &= \frac{1}{\pi^n \operatorname{vol}(E)} \int_{\mathbb{R}^n} \int_{s \geq ||t||_{E^\circ}} e^{-s} ds\, d\mu(t) \\ &= \frac{1}{\pi^n \operatorname{vol}(E)} \int_0^\infty e^{-s} \int_{\{t\in\mathbb{R}^n : ||t||_{E^\circ} \leq s\}} d\mu(t)\, ds \\ &= \frac{\operatorname{vol}(E^\circ)}{\pi^n \operatorname{vol}(E)} \int_0^\infty s^n e^{-s} ds = \frac{n! \operatorname{vol}(E^\circ)}{\pi^n \operatorname{vol}(E)}. \end{aligned} \tag{2.3}$$

Now, we return to D. Fix a positive $\delta \ll \operatorname{vol}(D)$. For each $v \in S^{n-1}$, let r_v denote the unique real number such that

$$\operatorname{vol}(\{x \in D : x \cdot v > r_v\}) = \delta.$$

Set $H_v := \{x \in \mathbb{R}^n : x \cdot v = r_v\}$ and $D|_v := \{x \in D : x \cdot v > r_v\}$. $D|_v$ is a continuous family of convex domains in D, each of volume δ. We let E_v denote the circumscribed Löwner-John ellipsoid of $D|_v$ — i.e., the unique ellipsoid of minimal volume that contains $D|_v$ (see [1, Lecture 3], for more on Löwner-John ellipsoids). Then, due to a result by F. John ([7]), if

$$E_v = c_v + A_v(\mathbb{B}^n),$$

for some $A_v \in \mathrm{GL}(n;\mathbb{R})$, then on shrinking,

$$E_v^n := c_v + \frac{1}{n} A_v(\mathbb{B}_n) \subseteq D|_v.$$

In particular, for every $v \in S^{n-1}$,

$$\operatorname{vol}(E_v^n) = \frac{1}{n^n} \operatorname{vol}(E_v) \geq \frac{1}{n^n} \operatorname{vol}(D|_v) = \frac{\delta}{n^n}. \tag{2.4}$$

We now estimate the Bergman kernel of D at each c_v. We first observe that since the Bergman kernel is invariant under translations, $K_{\frac{1}{n}A_v(\mathbb{B}^n)}(0) = K_{E_v^n}(c_v)$ for each $v \in S^{n-1}$. But, since $\frac{1}{n}A_v(\mathbb{B}^n)$ is an origin-symmetric convex domain in $\mathbb{R}^n$, we get by (2.3) that

$$K_{E_v^n}(c_v) = K_{\frac{1}{n}A_v(\mathbb{B}^n)}(0) \leq \frac{n!\,\mathrm{vol}\left(\left(\frac{1}{n}A_v(\mathbb{B}^n)\right)^\circ\right)}{\pi^n\,\mathrm{vol}\left(\frac{1}{n}A_v(\mathbb{B}^n)\right)}. \tag{2.5}$$

This can be combined with the Blaschke-Santaló inequality for origin-symmetric convex bodies:

$$\mathrm{vol}(D^\circ)\,\mathrm{vol}(D) \leq (\omega_n)^2,$$

and (2.4), to obtain that

$$K_{E_v^n}(c_v) \leq \frac{n!(\omega_n)^2}{\pi^n\,\mathrm{vol}\left(\frac{1}{n}A_v(\mathbb{B}^n)\right)^2} = \frac{n!(\omega_n)^2}{\pi^n\,\mathrm{vol}(E_v^n)^2} \leq \frac{n!n^{2n}(\omega_n)^2}{\pi^n\delta^2}.$$

Since $c_v \in E_v^n \subseteq D|_v \subset D$, by the monotonicity of the Bergman kernel,

$$K_D(c_v) \leq K_{E_v^n}(c_v) \leq u_n\delta^{-2}, \qquad \text{for every } v \in S^{n-1}, \tag{2.6}$$

where $u_n := \dfrac{n!n^{2n}(\omega_n)^2}{\pi^n}$.

Now, we claim that the image of the map $\gamma : S^{n-1} \to D \setminus D_\delta$ given by $v \mapsto c_v$ 'surrounds' D_δ — i.e., D_δ is contained in an open set U such that $\partial U \subseteq \gamma(S^{n-1})$. Our argument is as follows. Let b_v denote the barycenter of $H_v \cap D$. Then, by Lemma 2 in [14], every $x \in \partial D_\delta$ coincides with a b_v for some $v \in S^{n-1}$. Thus, the image of the map $\beta : S^{n-1} \mapsto D \setminus D_\delta$ given by $v \mapsto b_v$ surrounds D_δ (in the sense described above — in fact, $U = D_\delta$ in this case). Now, $T : S^{n-1} \times [0,1] \mapsto D \setminus D_\delta$ given by $(v,t) \mapsto (1-t)b_v + tc_v$ is a homotopy between $\beta(S^{n-1})$ and $\gamma(S^{n-1})$ whose image is entirely contained in the complement of D_δ. Thus, $\gamma(S^{n-1})$ must surround D_δ as well, and there is an open set $U \subset D$, such that $U \supseteq D_\delta$ and $\partial U \subseteq \gamma(S^{n-1})$. Thus, by the maximum principle ($x \mapsto \log K_D(x)$ is strongly convex on D),

$$\sup_{x \in D_\delta} K_D(x) \leq \sup_{y \in U} K_D(y) \leq \sup_{y \in \partial U} K_D(y) \leq \sup_{v \in S^{n-1}} K_D(c_v) \leq \frac{u_n}{\delta^2}.$$

This shows that $D_\delta \subset D^{u_n\delta^{-2}}$.

We now turn to the existence of ℓ_n. Once again, we fix δ so small that D_δ is nonempty, and H_v is as before. It suffices to show that for any $x \in H_v \cap D$, $K_D(x) \geq \ell_n\delta^{-2}$ for some $\ell_n > 0$ independent of v, δ and D. This is because for any $x \in \partial D_\delta$, there is a supporting hyperplane of D_δ that cuts off a set of volume δ from D — i.e., there is a $v \in S^{n-1}$ such that $x \in H_v \cap D$ (see Lemma 2 in [14]). The required estimate will be obtained from the following lower bound for convex domains due to Błocki in [2]:

$$K_\Omega(w,w) \geq \frac{1}{\mathrm{vol}_{\mathbb{C}^n}(I_\Omega(w))}, \qquad w \in \Omega, \tag{2.7}$$

where $I_\Omega(w) \subset \mathbb{C}^n$ is the Kobayashi indicatrix of Ω given by

$$I_\Omega(w) = \{\phi'(0) : \phi \in \mathcal{O}(\mathbb{D};\Omega), \phi(0) = w\}.$$

For us, $\Omega := \mathbb{R}^n + iD$, and $w = ix$ for some $x \in H_v \cap D$. We are seeking an upper bound on $\mathrm{vol}_{\mathbb{C}^n}(I_\Omega(w))$.

Without loss of generality, we assume that x is the origin in $\mathbb{R}^n$ and $v = (0, \ldots, 0, 1)$. In particular, H_v is the hyperplane $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$ and $D \cap \{x_n > 0\} = D|_v$. We will follow Nazarov's technique from [8] (as used by Błocki in [2]). We recall that $D^\circ = \{u \in \mathbb{R}^n : y \cdot u \leq 1 \text{ for all } y \in D\}$, which is the same as $\{u \in \mathbb{R}^n : y \cdot u < 1 \text{ for all } y \in D\}$ since D is open. Now, consider the half-plane $S := \{z \in \mathbb{C} : \operatorname{Im} z < 1\}$, and let $\Phi : S \mapsto \mathbb{D}$ denote the conformal map $z \mapsto -iz/(z - 2i)$. Then, $\Phi(0) = 0$ and $\Phi'(0) = 1/2$. For a fixed $u \in D^\circ$ and any $\phi \in \mathcal{O}(\mathbb{D}; \Omega)$ such that $\phi(0) = w$, the map $F : z \mapsto \Phi(\phi(z) \cdot u)$ is a holomorphic self-map of $\mathbb{D}$ that fixes the origin (since we are assuming that w is the origin in $\mathbb{C}^n$). Thus, by the Schwarz lemma, $|F'(0)| \leq 1$, or $|\phi'(0) \cdot u| \leq 2$. So, $\frac{1}{2} I_\Omega(w) \subseteq D_{\mathbb{C}}$, where

$$D_{\mathbb{C}} := \{z \in \mathbb{C}^n : |z \cdot u| \leq 1 \text{ for all } u \in D^\circ\}.$$

Note that $D_{\mathbb{C}} \subseteq \Big(\hat{D} \cup (-\hat{D})\Big) + i\Big(\hat{D} \cup (-\hat{D})\Big)$, where

$$\hat{D} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x \cdot u| \leq 1 \text{ for all } u \in D^\circ, x_n \geq 0\}.$$

But,

$$\hat{D} \subseteq \overline{D} \cap \{x \in \mathbb{R}^n : x_n \geq 0\} \subseteq \overline{D|_v},$$

and $\operatorname{vol}(D|_v) = \delta$. Thus, recalling (2.7),

$$K_D(x) = K_\Omega(w, w) \geq \frac{1}{\operatorname{vol}_{\mathbb{C}^n}(I_\Omega(w))} \geq (2)^{-2n} (2\delta)^{-2}.$$

Therefore, $D_\delta \supseteq D^{\ell_n \delta^{-2}}$, where $\ell_n := \dfrac{1}{4^{n+1}}$. This completes the proof of Theorem 1.1. ∎

3. A new affine invariant and some examples

It is unlikely that the values of ℓ_n and u_n computed above are optimal. For one, John's theorem on Löwner-John ellipsoids can be replaced by results that utilize other centrally-symmetric bodies, perhaps yielding better bounds. However, we believe optimal bounds can be obtained if we restrict ourselves to certain classes of convex bodies. Before we support this claim with some computations, we associate a new quantity θ_D to a convex body.

Definition. Suppose $D \subset \mathbb{R}^n$ is a convex body. Let

$$\begin{aligned} \ell_D &:= \liminf_{\delta \to 0} \Big(\sup\{\ell > 0 : D^{\ell/\delta^2} \subseteq D_\delta\}\Big); \\ u_D &:= \limsup_{\delta \to 0} \Big(\inf\{u > 0 : D_\delta \subseteq D^{u/\delta^2}\}\Big); \\ \theta_D &:= \frac{\ell_D}{u_D}. \end{aligned}$$

We establish some properties of θ_D.

Proposition 3.1. *For a convex body $D \subset \mathbb{R}^n$,*

(1) $\dfrac{\pi^n}{n! n^{2n} 4^{n+1} (\omega_n)^2} \leq \theta_D \leq 1.$

(2) *θ_D is affine invariant, i.e., $\theta_D = \theta_{A(D)}$ for any affine map A on $\mathbb{R}^n$.*

Proof. (1) The upper bound on θ_D follows from the fact that $\ell_D \leq u_D$, by definition. The lower bound is a consequence of Theorem 1.1, where we have essentially shown that $\ell_D \geq 1/4^{n+1}$ and $u_D \leq n!n^{2n}(\omega_n)^2/\pi^n$.

(2) The affine invariance of θ_D follows from that of ℓ_D and u_D, which, in turn, is a consequence of the transformation properties of D_δ and D^M under affine maps. More concretely, if $A : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map and H is a hyperplane that cuts off a set of volume δ from D, then the hyperplane $A(H)$ cuts off a set of volume $|\det(A)|\delta$ from $A(D)$. Therefore,

$$A(D_\delta) = A(D)_{|\det A|\delta}, \qquad \text{for all } \delta > 0. \tag{3.1}$$

Now, let $\Omega := \mathbb{R}^n + iD$ and $A_{\mathbb{C}}$ be the map $z \mapsto Az$. Then, $A_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphic map with $\operatorname{Jac}_{\mathbb{C}} A_{\mathbb{C}} = \det A$, where $\operatorname{Jac}_{\mathbb{C}}$ denotes the complex Jacobian. We use the well-known fact that the Bergman kernel of Ω satisfies

$$K_{A_{\mathbb{C}}(\Omega)}(A_{\mathbb{C}}(z), A_{\mathbb{C}}(z))|\operatorname{Jac}_{\mathbb{C}}(A_{\mathbb{C}})|^2 = K_\Omega(z,z), \qquad \text{for all } z \in \Omega.$$

Hence,

$$A(D^M) = A(D)^{M/|\det A|^2}. \tag{3.2}$$

Combining (3.1) and (3.2), we see that if $D^{\ell/\delta^2} \subseteq D_\delta \subseteq D^{u/\delta^2}$, then $A(D)^{\ell/(|\det A|\delta)^2} \subseteq D_{|\det A|\delta} \subseteq D^{u/(|\det A|\delta)^2}$. Hence, the affine invariance of ℓ_D, u_D and θ_D. □

We now compute some examples to indicate the extent to which θ_D distinguishes convex domains.

Proposition 3.2. *If D is strongly convex — i.e., the second fundamental form on ∂D is positive definite everywhere on ∂D — then, $\theta_D = 1$.*

Proof. We begin with some notation (see Figure 1). For $x \in \partial D$, let $N(x)$ be the unique outer unit normal to ∂D at x, and $H(x) = \{y \in \mathbb{R}^n : y\cdot N(x) = x\cdot N(x)\}$. For $\delta > 0$, let $\Delta(x,\delta)$ denote the width of the slice of volume δ cut off by a hyperplane $H(x,\delta)$ perpendicular to $N(x)$ — i.e.,

$$\operatorname{vol}\{y \in D : y \cdot N(x) > x \cdot N(x) - \Delta(x,\delta)\} = \delta$$

and

$$\begin{aligned} H(x,\delta) &= \{y \in \mathbb{R}^n : y \cdot N(x) = x \cdot N(x) - \Delta(x,\delta)\} \\ &= H(x) - \Delta(x,\delta)N(x). \end{aligned}$$

Let x^δ denote the barycenter of $H(x,\delta) \cap D$.

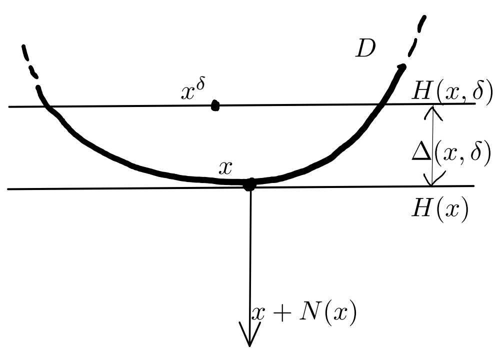

FIGURE 1

Now — as θ_D is affine invariant — for a fixed $x_0 \in \partial D$, we can choose affine co-ordinates, so that x_0 is the origin, the outer unit normal $N(x_0) = (0, \ldots, 0, -1)$ and $H(x_0, \delta) = \{(x', y) : y = \Delta(x_0, \delta)\}$, where $x' = (x_1, \ldots, x_{n-1})$. There is a neighborhood U_0 of x_0 such that $U_0 \cap D = \{y > \phi(x')\}$, where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a convex function of the form

$$\phi(x) = \alpha(x_1^2 + \cdots + x_{n-1}^2) + \text{h. o. t.},$$

for some $\alpha > 0$. Thus, each $H(x_0, \delta) \cap D$ satisfies the equation

$$\Delta(x_0, \delta) = \alpha(x_1^2 + \cdots + x_{n-1}^2) + \text{h. o. t.}$$

in the hyperplane $y = \Delta(x_0, \delta)$. So, we may estimate the barycenter of $H(x_0, \delta) \cap D$ as

$$x_0^\delta = (o(\sqrt{\Delta(x_0, \delta)}), \ldots, o(\sqrt{\Delta(x_0, \delta)}), \Delta(x_0, \delta)) \qquad \text{as } \delta \to 0.$$

Thus, minimizing $\operatorname{dist}(x_0^\delta, z)$ over all $z \in \partial\Omega$, we obtain that

$$\lim_{\delta \to 0} \frac{\Delta(x_0, \delta)}{\operatorname{dist}(x_0^\delta, \partial D)} = 1. \tag{3.3}$$

Moreover, using Dupin indicatrices (see [12, Lemma 10]), it is known that

$$\lim_{\delta \to 0} \frac{\Delta(x_0, \delta)^{n+1}}{\delta^2} = \frac{1}{2^{n+1}} \left(\frac{n+1}{\omega_{n-1}} \right)^2 \kappa(x_0), \tag{3.4}$$

where κ is the Gaussian curvature function of ∂D. Lastly, since Ω is strongly convex, $\Omega = \mathbb{R}^n + iD$ is strongly pseudoconvex. Thus, by Hörmander's estimate (in [6]), we have that

$$\lim_{x \to x_0 \in \partial D} \operatorname{dist}(x, \partial D)^{n+1} K_D(x) = \frac{n!}{(4\pi)^n} \kappa(x_0). \tag{3.5}$$

Since, $\lim_{\delta \to 0} x_0^\delta = x_0$, we can combine (3.3), (3.4) and (3.5) to obtain that

$$\lim_{\delta \to 0} \delta^2 K_D(x_0^\delta) = \frac{n! 2^{n+1}}{(4\pi)^n} \left(\frac{\omega_{n-1}}{n+1} \right)^2 =: a_n.$$

Hence, $(x_0, \delta) \mapsto \delta^2 K_D(x_0^\delta)$ extends to a (uniformly) continuous function on $\partial D \times [\hat{\delta}, 0]$. So, given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that for $\delta < \delta_\varepsilon$,

$$\frac{a_n - \varepsilon}{\delta^2} < K_D(x^\delta) < \frac{a_n + \varepsilon}{\delta^2}, \qquad \text{for all } x \in \partial D.$$

According to Lemma 2 in [14], each $y \in \partial D_\delta$ is the barycenter x^δ of some $H(x, \delta) \cap D$. Therefore, for $\delta < \delta_\varepsilon$,

$$D^{(a_n - \varepsilon)\delta^{-2}} \subset D_\delta \subset D^{(a_n + \varepsilon)\delta^{-2}}.$$

Thus,

$$\theta_D = \frac{\liminf_{\delta \to 0} \sup\{\ell > 0 : D^{\ell/\delta^2} \subseteq D_\delta\}}{\limsup_{\delta \to 0} \inf\{u > 0 : D_\delta \subseteq D^{u/\delta^2}\}} \geq \frac{a_n - \varepsilon}{a_n + \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, and $\theta_D \leq 1$, our claim follows. □

We contrast the above example with the next one, where the Gaussian curvature of the boundary vanishes on a large part of it.

Proposition 3.3. *Let $D \Subset \mathbb{R}^2$ be a triangle or a parallelogram. Then, $\theta_D = 4/\pi^2$.*

Proof. As all planar triangles and parallelograms are affine images of the triangle $T = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}$ and the square $S := (0, 1) \times (0, 1)$, respectively, it suffices to show that $\ell_S = \ell_T$, $u_S = u_T$ and $\theta_S = 4/\pi^2$. We take this approach as it is hard to directly compute θ_T.

We start with a description of the floating body of S. For small enough $\delta > 0$, the boundary of S_δ is a piecewise smooth curve, each smooth piece of which is a part of a hyperbola (see Figure 2). Specifically,

$$S_\delta = \Big\{(x, y) \in \mathbb{R}^2 : \min\Big(xy, (1 - x)y, x(1 - y), (1 - x)(1 - y)\Big) > \delta/2\Big\}.$$

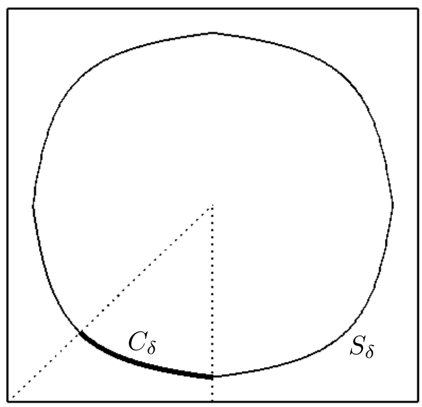

FIGURE 2. A convex floating body for S.

Due to the eight-fold symmetry of S, we will focus on the one-eighth part of the boundary given by $C_\delta := \partial S_\delta \cap \{(x, y) : 0 \le y \le x \le 1/2\}$ (thickened in Figure 2). For $\delta \ll 1/2$, C_δ can be parametrized as

$$t \mapsto c(t) := \left(t, \frac{\delta}{2t}\right), \qquad \sqrt{\frac{\delta}{2}} \le t \le \frac{1}{2}.$$

To estimate K_S on C_δ, we observe that

$$C_\delta \subset T \subset S \subset \widetilde{T} \qquad \text{(see Figure 3)},$$

where $\widetilde{T}$ is the image of T under the map $(x, y) \mapsto (2x, 2y)$.

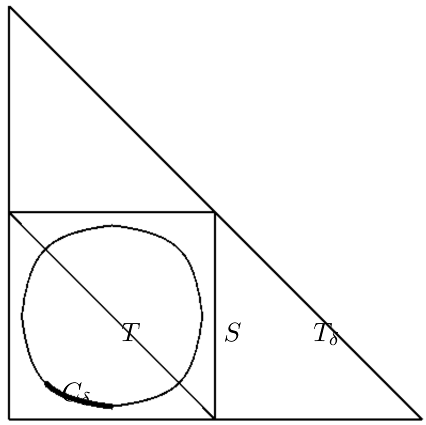

FIGURE 3

The descriptions of the floating bodies of T and $\widetilde{T}$ are also needed:

$$\begin{aligned} T_\delta &= \left\{(x,y)\in\mathbb{R}^2 : \min\Big(xy,(1-x-y)y,(1-x-y)x\Big) > \delta/2\right\}; \\ \widetilde{T}_\delta &= \left\{(x,y)\in\mathbb{R}^2 : \min\Big(xy,(2-x-y)y,(2-x-y)x\Big) > \delta/2\right\}. \end{aligned}$$

These explicit descriptions allow us to conclude that, for $\delta << 1/2$,

$$C_\delta \subset T_{\delta-2\delta^2} \quad \text{and} \quad C_\delta \subset \partial\widetilde{T}_\delta \subset \widetilde{T}\setminus\widetilde{T}_{\delta+2\delta^2}. \tag{3.6}$$

Now fix an arbitrary $\varepsilon > 0$. Then, for small enough δ,

(1) $(1-\varepsilon)\delta < \delta - 2\delta^2$ and $\delta + 2\delta^2 < (1+\varepsilon)\delta$; and
(2) $T_\delta \subseteq T^{(1+\varepsilon)u_T\delta^{-2}}$ and $\widetilde{T}^{(1-\varepsilon)\ell_T\delta^{-2}} \subseteq \widetilde{T}_\delta$.

The latter follows from the definitions of ℓ_D and u_D, and the fact that $\ell_{\widetilde{T}} = \ell_T$ due to affine invariance (established in the proof of Proposition 3.1). We combine (3.6), (1), the monotonicity of T_δ and $\widetilde{T}_\delta$, and (2) to conclude that:

$$C_\delta \subset T_{\delta-2\delta^2} \subset T_{(1-\varepsilon)\delta} \subset T^{(1+\varepsilon)u_T(1-\varepsilon)^{-2}\delta^{-2}}$$

and

$$C_\delta \subset \widetilde{T}\setminus\widetilde{T}_{\delta+2\delta^2} \subset \widetilde{T}\setminus\widetilde{T}_{(1+\varepsilon)\delta} \subset \widetilde{T}\setminus\widetilde{T}^{(1-\varepsilon)\ell_T(1+\varepsilon)^{-2}\delta^{-2}}.$$

Thus, for all $c \in C_\delta$,

$$K_T(c) < \frac{(1+\varepsilon)u_T}{(1-\varepsilon)^2\delta^2} \quad \text{and} \quad K_{\widetilde{T}}(c) > \frac{(1-\varepsilon)\ell_T}{(1+\varepsilon)^2\delta^2}.$$

So, by the monotonocity of the Bergman kernel,

$$\frac{(1-\varepsilon)\ell_T}{(1+\varepsilon)^2\delta^2} < K_{\widetilde{T}}(c) < K_S(c) < K_T(c) < \frac{(1+\varepsilon)u_T}{(1-\varepsilon)^2\delta^2}.$$

As $\varepsilon > 0$ was arbitrarily chosen, and the estimates on C_δ transfer to ∂S_δ due to symmetry,

$$S^{\ell_T\delta^{-2}} \subset S_\delta \subset S^{u_T\delta^{-2}}.$$

Thus, $u_S \leq u_T$, $\ell_S \geq \ell_T$ and, consequently, $\theta_S \geq \theta_T$. An analogous computation can be executed after switching the roles of S and T to obtain that $\theta_T \geq \theta_S$, thus yielding the desired equality. It now suffices to compute θ_S.

We use (1.2) to compute the Bergman kernel of $\mathbb{R}^2 + iS$ at any point $(x, y) \in S$:

$$K_S\big((x,y)\big) = \frac{\pi^2}{16}\csc^2(\pi x)\csc^2(\pi y).$$

Once again, we can exploit the symmetry of S to obtain that

$$\ell_S = \lim_{\delta\to 0}\inf_{c\in C_\delta} K_S(c)\delta^2 = \lim_{\delta\to 0}\inf_{t\in[\sqrt{\delta/2},1/2]} \frac{\pi^2\delta^2}{16}\csc^2(\pi t)\csc^2\left(\frac{\pi\delta}{2t}\right) = \frac{1}{4\pi^2};$$

$$u_S := \lim_{\delta\to 0}\sup_{c\in C_\delta} K_S(c)\delta^2 = \lim_{\delta\to 0}\sup_{t\in[\sqrt{\delta/2},1/2]} \frac{\pi^2\delta^2}{16}\csc^2(\pi t)\csc^2\left(\frac{\pi\delta}{2t}\right) = \frac{1}{16}.$$

Therefore, $\theta_T = \theta_S = 4/\pi^2$. □

We strongly suspect that $\theta_D = 1$ completely characterizes strongly convex bodies, and that Proposition 3.3 can be extended to all planar convex polygons. In fact, we believe that, for $n = 2$, these represent the two extremes of the range of values for θ_D (this would improve the first part of Proposition 3.1). Furthermore, it is likely that using the almost polygonal bodies constructed in [13] one can construct planar convex bodies with any prescribed value of θ_D in the interval $(4/\pi^2, 1)$.

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