UM 101 HOMEWORK ASSIGNMENT 1

Posted on October 20, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study. You must first try these **on your own** before seeking hints from the instructor/TAs.
- Some of these problems will be discussed at the next tutorial. The TA will not give complete solutions, but will provide hints.
- A 10-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

Problem 1. Let A be a Peano set and S be the successor function on A (as defined in the first lecture). Show, using only the axioms of Peano, that the range of S is $A \setminus \{0\}$. For this questios, please interpret the words "function" and "range" in the way you did in school, and not in the set-theoretic way introduced in class.

Problem 2. We mentioned in class that when listing the ZFC axioms, we do not need to add additional axioms for the existence of the intersection or the set-difference of two sets. Using the ZFC axioms, prove the following statements.

(a) Given two sets A and B, show that $A \cap B$ exists as a set.
(b) Given two sets A and B, show that $A \setminus B$ exists as a set.

Problem 3. Given two objects a, b, let (a, b) denote the set $\{\{a\}, \{a, b\}\}$. First argue why the ZFC axioms guarantee the existence of this set. Then, show that $(a, b) = (c, d)$(as sets) if and only if $a = c$ and $b = d$.

Problem 4. Prove Lemma 1.4. I.e., show that if $\mathscr{C}$ is a non-empty set of inductive sets, then

$$\bigcap_{A \in \mathscr{C}} A$$

is an inductive set.

Problem 5. Let A, B, C, D be sets. Some of the following statements are always true, and the others are sometimes wrong. Decide which is which. For the ones you declare "always true", provide a proof. For the others, provide one counterexample each.

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
(b) $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$.
(c) $C \cap (A \setminus B) = A \cap (C \setminus B)$.
(d) $C \cup (A \setminus B) = A \cup (C \setminus B)$.

Problem 6. Let A be a set. Define a relation $\mathbf{R}$ such that for any subsets B and C of A,

$$B \,\mathbf{R}\, C \iff B \subseteq C.$$

Remember that a relation $\mathbf{R}$ is a subset of a Cartesian product of sets. Is the relation that you've defined a function?

Problem 7. From the definition of $+$ and $\cdot$ on $\mathbb{N}$ (as defined in class), prove that for all $m, n, k \in \mathbb{N}$,

$$m \cdot (n + k) = (m \cdot n) + (m \cdot k).$$