UM 101 HOMEWORK ASSIGNMENT 1

SKETCH OF SOLUTIONS

Problem 1. Let A be a Peano set and S be the successor function on A (as defined in the first lecture). Show, using only the axioms of Peano, that the range of S is $A \setminus \{0\}$. For this questios, please interpret the words "function" and "range" in the way you did in school, and not in the set-theoretic way introduced in class.

Proof. Let R be the range of S. By the **Axiom P2**, S is function from A to itself. Thus, $R \subseteq A$, and we may write

$$R = \{b \in A : b = S(a) \text{ for some } a \in A\}.$$

By **Axiom P3**, $S(a) \neq 0$ for any $a \in A$. Thus,

$$R \subseteq A \setminus \{0\}. \tag{1}$$

Now, let $B = R \cup \{0\}$. Note that $B \subseteq A$, $0 \in B$, and for any $a \in B$, $S(a) \in B$. The last statement holds because if $a \in B$, then $a \in A$, and therefore, by definition, $S(a) \in R \subseteq B$. By **Axiom P5**,

$$B = R \cup \{0\} = A.$$

Thus, if $x \in A$, then either $x \in R$ or $x = 0$, but it cannot be both since $R \cap \{0\} = \emptyset$. Thus,

$$A \setminus \{0\} \subseteq R. \tag{2}$$

Combining (1) and (2) gives the claim. □

Problem 2. We mentioned in class that when listing the ZFC axioms, we do not need to add additional axioms for the existence of the intersection or the set-difference of two sets. Using the ZFC axioms, prove the following statements.

(a) Given two sets A and B, show that $A \cap B$ exists as a set.
(b) Given two sets A and B, show that $A \setminus B$ exists as a set.

Proof. (a) We are given two sets A and B. By Axiom of specification, we can consider

$$C = \{x \in A : P(x) \text{ is true}\}$$

where $P(x)$ is true if and only if $x \in B$. Now, C will be a set, and by the definition of intersection $C = A \cap B$.

(b) Again by Axiom of specification,

$$D = \{x \in A : P^{'}(x) \text{ is true}\}$$

where $P^{'}(x)$ is true if and only if $x \notin B$. Here agian D will also be a set, and by the definition of set difference, $D = A \setminus B$. □

Problem 3. Given two objects a, b, let (a, b) denote the set $\{\{a\}, \{a, b\}\}$. First argue why the ZFC axioms guarantee the existence of this set. Then, show that $(a, b) = (c, d)$(as sets) if and only if $a = c$ and $b = d$.

Only proof of $(a, b) = (c, d) \Rightarrow a = c$ *and* $b = d$. Case I. $a = b$. In this case, by the axiom of extension (AoE), $\{a, b\} = \{a\}$, and thus, $(a, b) = \{\{a\}\}$. Thus, by AoE, if $x \in (c, d)$, then $x = \{a\}$. So, $\{c\} = \{a\}$ and $\{c, d\} = \{a\}$. Once again, by AoE, $c = d = a$. Thus, $a = c$ and $b = a = c = d$.

Case II. $a \neq b$. Since $(a, b) = (c, d)$, by the axiom of extension, any element of (a, b) is an element of (c, d). Thus, $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$ or $\{a\} = \{c, d\}$ and $\{a, b\} = \{c\}$. In the first case, $a = c$, by AoE. Now, since any element of $\{a, b\}$ must be an element of $\{c, d\}$ and $b \neq a = c$, it must be that $b = d$. In the second case, we get that $a = c = d$. Exchanging the roles of a with c, and b with d, we have returned to Case I for which we have already shown the conclusion.

Problem 4. Prove Lemma 1.4. I.e., show that if $\mathscr{C}$ is a non-empty set of inductive sets, then

$$\bigcap_{A \in \mathscr{C}} A$$

is an inductive set.

Main idea. If you consider $\mathcal{G} = \bigcap_{A \in \mathscr{C}} A$ then you need to show (**by using the definition of intersections and the definition of inductive sets**) that

1. $\emptyset \in \mathcal{G}$
2. For every $A \in \mathcal{G} \Rightarrow A^{+} \in \mathcal{G}$

Problem 5. Let A, B, C, D be sets. Some of the following statements are always true, and the others are sometimes wrong. Decide which is which. For the ones you declare "always true", provide a proof. For the others, provide one counterexample each.

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
(b) $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$.
(c) $C \cap (A \setminus B) = A \cap (C \setminus B)$.
(d) $C \cup (A \setminus B) = A \cup (C \setminus B)$.

Proof. **Part(a)** We will use the **Axiom of Extension** to show that both the sets are equal. So, we will show that

$$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \tag{3}$$

and

$$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \tag{4}$$

For (), let $(a, b) \in A \times (B \cup C)$ be any arbitrary element. So, $a \in A$ and $b \in B \cup C$. Which implies that

$$a \in A \text{ and either } b \in B \text{ or } b \in C$$

So,

$$a \in A \text{ and } b \in B \text{ or } a \in A \text{ and } b \in C \Rightarrow (a, b) \in (A \times B) \cup (A \times C).$$

This proves (3). Similarly you can prove (4).

Part(b). Here if we look at the set on left hand side, it contains elements of the form (a, b) where $a \in A, b \in B$ and either $a \notin C$ or $b \notin D$ or $a, b \notin A, B$, but the set on the right hand side contains the element of the form (a, b), where $a \in A, b \in B$ with $a \notin C$ and $b \notin B$. This indicated that both sets are not equal in general. So, we need to produce a counter example.

Counterexample. Let $A = \{1, 2\},\ B = \{3, 4\},\ C = \{2, 3\},\ D = \{4, 5\}$. In this example, you can see that $(1, 4) \in (A \times B) \setminus (C \times D)$ but $(1, 4) \notin A \setminus C \times B \setminus D$.

Parts (c) and (d). You can use Venn diagrams to get a sense of whether the two sets will be equal or not (make sure you are considering all possibilities in your diagrams). Then proceed like in Part (a) if proving the claim, or construct a counterexample like in Part (b). $\square$

Problem 6. Let A be a set. Define a relation $\mathbf{R}$ such that for any subsets B and C of A,

$$B \mathbf{R} C \iff B \subseteq C.$$

Remember that a relation $\mathbf{R}$ is a subset of a Cartesian product of sets. Is the relation that you've defined a function?

We have a set A. We know that if we have a set X and we are talking about a relation on X then it will be a subset R of $X \times X$ i.e. $R \subseteq X \times X$ and if $(a, b) \in R$ then we will write $a \ R \ b$.

In this problem, you need to define the relation $\mathbf{R}$ on subsets of A as

$$\text{for } B, C \subseteq A, B \ \mathbf{R} \ C \iff B \subseteq C$$

Which indicates that $\mathbf{R}$ should be a subset of $\mathscr{P}(A) \times \mathscr{P}(A)$, where $\mathscr{P}(A)$ is the power set of A because $B, C \in \mathscr{P}(A)$ only. So, must set

$$\mathbf{R} = \{(B, C) : B \subseteq C\} \subseteq \mathscr{P}(A) \times \mathscr{P}(A)$$

with $Dom(\mathbf{R}) \subseteq \mathscr{P}(A)$ and $Ran(\mathbf{R}) \subseteq \mathscr{P}(A)$.

This relation need not be a function. *Hint for this part.* a function from $\mathscr{P}(A)$ to $\mathscr{P}(A)$ is a relation i.e. a subset of $\mathscr{P}(A) \times \mathscr{P}(A)$, in which the domain is $\mathscr{P}(A)$ and each element of the domain related to at most one element in the range. Assume $A \neq \emptyset$. What are the elements that the $\emptyset$ is related to?

IMPORTANT NOTE: For the second part of the above problem, if you fix a $B \in \mathscr{P}(A)$, and try to determine all the pairs $(B, C) \in R$, you have to take ALL possible subsets C of A such that $B \subseteq C$. You cannot restrict your definition of the relation. E.g., you cannot say

$$(B, C) \in R \iff B = C.$$

This does not capture the given relation in the problem.

Problem 7. From the definition of $+$ and $\cdot$ on $\mathbb{N}$ (as defined in class), prove that for all $m, n, k \in \mathbb{N}$,

$$m \cdot (n + k) = (m \cdot n) + (m \cdot k).$$

Main Idea. Fix m and n and induct on k. It helps to write the claim as

$$\text{prod}_m(n + k) = \text{prod}_m(n) + \text{prod}_m(k).$$