UM 101 HOMEWORK ASSIGNMENT 2

Posted on October 27, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study. You must first try these **on your own** before seeking hints from the instructor/TAs.
- Some of these problems will be discussed at the next tutorial. The TA will not give complete solutions, but will provide hints.
- A 10-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

Problem 1. (a) Prove that for any $m, n \in \mathbb{N}$, exactly one of the following statements hold.

(i) $m = n$;
(ii) there is a $k \in \mathbb{N} \setminus \{0\}$ such that $m + k = n$;
(iii) there is a $k \in \mathbb{N} \setminus \{0\}$ such that $n + k = m$.

You may use: induction, the definition of sum_m and any of its six properties stated in class (as Theorem 1.12), and the fact that the range of the function $f(x) = x + 1$ on $\mathbb{N}$ is $\mathbb{N} \setminus \{0\}$ (Problem 1 in HW1).

(b) Show that $\mathbb{N}$ is an ordered set if we define $<$ as follows: $m < n$ if there is a $k \in \mathbb{N} \setminus \{0\}$ such that $m + k = n$.

Problem 2. Let $(F, +, \cdot)$ be a field. According to Axiom (F5), given $x \in F$, there is a $y \in F$ such that $x + y = 0$. Show that y is unique, i.e., if there is a $z \in F$ such that $x + y = x + z = 0$, then $y = z$. Use only the field axioms to justify your answer.

Problem 3. Let $+$ and $\cdot$ be the usual addition and multiplication on $\mathbb{N}$. You are free to use their well-known properties.
(a) Let $F = \{0, 1, 2, 3\}$. We endow F with addition and multiplication as follows.

$$a \oplus b = c, \quad \text{where } c \text{ is the remainder that } a + b \text{ leaves when divided by } 4,$$
$$a \odot b = d, \quad \text{where } d \text{ is the remainder that } a \cdot b \text{ leaves when divided by } 4.$$

Is $(F, \oplus, \odot)$ a field? Please justify your answer.

(b) Let $F = \{0, 1\}$. We endow F with addition and multiplication as follows.

$$a \oplus b = c, \quad \text{where } c \text{ is the remainder that } a + b \text{ leaves when divided by 2,}$$
$$a \odot b = d, \quad \text{where } d \text{ is the remainder that } a \cdot b \text{ leaves when divided by 2.}$$

You may assume $(F, \oplus, \odot)$ is a field (or treat this as an additional exercise, but this won't appear on your quiz). Is it possible to give F a relation $<$ so that $(F, \oplus, \odot, <)$ is an ordered field? Please justify your answer.

Problem 4. Let $(F, +, \cdot, <)$ be an ordered field.

(i) Using only the field axioms, and the uniqueness of the additive inverse, show that for all $a, b, c, \in F$, $a(b - c) = ab - ac$.

(ii) Using the field axioms, the order axioms, and Part (i), show that for all $a, b, c, \in F$, if $a < b$ and $c < 0$, then $bc < ac$.

Problem 5. Apostol defines an ordered field as a field $(F, +, \cdot)$ together with a set $P \subseteq F$ satisfying the following axioms.

(O′1) If $x, y \in P$, then $x + y \in P$ and $x \cdot y \in P$.

(O′2) For every $x \in F$ such that $x \neq 0$, either $x \in P$ or $-x \in P$, but not both.

(O′3) $0 \notin P$.

Show that our definition of an ordered field is equivalent to that of Apostol's. That is, show that for a field $(F, +, \cdot)$:

(i) if there is a relation $<$ satisfying (O1)-(O4), then there is a $P \subseteq F$ satisfying (O'1)-(O'3), and

(ii) if there is a $P \subseteq F$ satisfying (O'1)-(O'3), then there is a relation $<$ satisfying (O1)-(O4).