UM 101 HOMEWORK ASSIGNMENT 3

Posted on November 03, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study. You must first try these **on your own** before seeking hints from the instructor/TAs.
- Some of these problems will be discussed at the next tutorial. The TA will not give complete solutions, but will provide hints.
- A 10-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

Throughout this assignment, you may view $\mathbb{N}$ *as a subset of* $\mathbb{R}$. *The operations* $+$, $\cdot$ *and the relation* $<$ *on* $\mathbb{N}$ *are those coming from* $\mathbb{R}$. *We will no longer worry about the ZFC and Peano axioms.* $\mathbb{P}$ *is the set of positive natural numbers.*

Problem 1. Let $x \in \mathbb{R}$ such that $0 \leq x < \delta$ for every $\delta > 0$. Show that x must be 0. Explicitly state the field and order axioms that you are using.

You may take basic arithmetic and order properties of $\mathbb{R}$ for granted going forward.

Problem 2. Formulate definitions of the terms "bounded below set", "lower bound" and "greatest lower bound" for subsets of $\mathbb{R}$. Show that $\mathbb{Z}$ is neither bounded above nor bounded below.
Note. You may not use (without proof) anything listed as a theorem in Sections 13.8 and 13.9 of Apostol's book.

Problem 3. If x is an arbitrary real number, prove that there is exactly one integer n which satisfies

$$n \leq x < n + 1.$$

You may use Theorem 1.28 from Apostol (without proof), which says $\mathbb{P}$ is not bounded above. Other than the least upper bound property of $\mathbb{R}$, you need not specify which axioms you are using in your proof. *Hint.* Consider the set $S = \{n \in \mathbb{Z} : n \leq x\}$.

Problem 4. Let $\{a_n\} \subset \mathbb{R}$ be an arbitrary sequence. Among the statements listed below, exactly one implies that $\{a_n\}$ is convergent, exactly one implies that $\{a_n\}$ is divergent, and the remaining one does not say anything conclusive about the convergence of $\{a_n\}$. Determine which is which. For the conclusive statements, you must give proofs. For the inconclusive statement, you must provide two sequences which satisfy the given statement, but one converges and the other diverges.

(1) There exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < n\varepsilon$ for all $n \geq N$.

(2) There exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \dfrac{\varepsilon}{n+1}$ for all $n \geq N$.

(3) For every $R > 0$, there exists an $N \in \mathbb{N}$ such that $|a_N| > R$

Note. As part of the above problem, you have established the following resullt: *every convergent sequence is bounded.*

Bonus (not for quiz). Are any of the above statements actually *equivalent* to the definition of convergence or divergence?

Problem 5. Determine which of the following sequences converge and which diverge. In the case of convergence, determine the limit.

(1) $\left\{\dfrac{2 - 3n^2}{n^2 + 2n + 1}\right\}_{n\in\mathbb{N}}$

(2) $\left\{\dfrac{3n^2 - 2}{3n + 1}\right\}_{n\in\mathbb{N}}$

(3) $\left\{n - \sqrt{1 + n^2}\right\}_{n\in\mathbb{N}}$

(4) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n\in\mathbb{N}}$

Note. You may use any of the theorems stated in Lectures 07 and 08, or established in this homework, but you must state what you are using.