UM 101 HOMEWORK ASSIGNMENT 3
SKETCH OF SOLUTIONS

Problem 1. Let $x \in \mathbb{R}$ such that $0 \leq x < \delta$ for every $\delta > 0$. Show that x must be 0. Explicitly state the field and order axioms that you are using.

Proof. Since $x \geq 0$, either $x = 0$ or $x > 0$. In the former case, there is nothing to prove. In the latter case, let $\delta = x$. Then, the condition on x says that $x < x$, which violates the trichotomy law O1. □

Problem 2. Formulate definitions of the terms "bounded below set", "lower bound" and "greatest lower bound" for subsets of $\mathbb{R}$. Show that $\mathbb{Z}$ is neither bounded above nor bounded below.

Note. You may not use (without proof) anything listed as a theorem in Sections 13.8 and 13.9 of Apostol's book.

For the definitions, see Apostol.

Proof of claim. To prove that $\mathbb{Z}$ is not bounded above , proceed just like in the proof of Theorem 1.28 in Apostol. To prove that $\mathbb{Z}$ is not bounded below, either

(a) proceed analogously using the fact that if $n \in \mathbb{Z}$, then $n - 1 \in \mathbb{Z}$. You must prove the fact that every (nonempty) bounded below set has an infimum. Or,

(b) you may directly argue that if $\mathbb{Z}$ is bounded below, i.e., if there is a $b \in \mathbb{R}$ such that $b \leq a$ for all $a \in \mathbb{Z}$, then $-b$ must be an upper bound of $\mathbb{Z}$, contradicting the first part.

Problem 3. If x is an arbitrary real number, prove that there is exactly one integer n which satisfies

$$n \leq x < n + 1.$$

You may use Theorem 1.28 from Apostol (without proof), which says $\mathbb{P}$ is not bounded above. Other than the least upper bound property of $\mathbb{R}$, you need not specify which axioms you are using in your proof. *Hint.* Consider the set $S = \{n \in \mathbb{Z} : n \leq x\}$.

Proof. □

Consider the set $S = \{n \in \mathbb{Z} : n \leq x\}$.

Step 1: Show that the set is S is nonempty. Use the fact that the set of all integers is not bounded below.

Step 2: Notice that the set S is bounded above by x. Now by the least upper bound property

S has a supremum say α. Since α is a supremum, there exists a $n_0 \in \mathbb{Z}$ such that $\alpha - 1 < n_0 \leq \alpha \leq x$. Now using the method of contradiction show that $x \leq n_0 + 1$.
Step 3: Show that the n_0 we obtained above is unique using the method of contradiction and the fact that given two integers m and n, $m < n$ if and only if there exists $k \in \mathbb{N} \setminus 0$ such that $m = n + k$.

Problem 4. Let $\{a_n\} \subset \mathbb{R}$ be an arbitrary sequence. Among the statements listed below, exactly one implies that $\{a_n\}$ is convergent, exactly one implies that $\{a_n\}$ is divergent, and the remaining one does not say anything conclusive about the convergence of $\{a_n\}$. Determine which is which. For the conclusive statements, you must give proofs. For the inconclusive statement, you must provide two sequences which satisfy the given statement, but one converges and the other diverges.

(1) There exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < n\varepsilon$ for all $n \geq N$.

Claim. This statement does not yield a conclusive result on the convergence of $\{a_n\}$.

Proof. Let $a_n = 1$ for all $n \in \mathbb{N}$. $\{a_n\}$ is convergent and satisfies the statement. On the other hand, $\{\sqrt{n}\}$ is divergent, but also satisfies the statement ($L = 0$). □

(2) There exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \dfrac{\varepsilon}{n+1}$ for all $n \geq N$.

Claim. $\{a_n\}$ is convergent.

Proof. Let $\varepsilon > 0$. Then, by the given statement, there is an $N \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon/(n+1) \quad \forall n \geq N.$$

Since $1/(n+1) < 1$ for all $n \in \mathbb{P}$, we have that

$$|a_n - L| < \varepsilon \quad \forall n \geq N.$$

This is the definition of convergence of $\{a_n\}$. □

(3) For every $R > 0$, there exists an $N \in \mathbb{N}$ such that $|a_N| > R$

Claim. $\{a_n\}$ is divergent.

Proof. Suppose not and our sequence is convergent to a limit say L. So given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for every $n \geq \mathbb{N}$. Now by triangular inequality we have that

$$|a_n| = |a_n - L + l| \leq |a_n - L| + |L| < \varepsilon + |L| \quad \forall n \geq N.$$

Now let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, \varepsilon + |L|\}$. Then we get that $|a_n| \leq M \quad \forall n \in \mathbb{N}$. i.e. we get that $\{a_n\}$ is bounded. A contradiction to our hypothesis. □

Note. As part of the above problem, you have established the following resullt: *every convergent sequence is bounded.*

Bonus (not for quiz). Are any of the above statements actually *equivalent* to the definition of convergence or divergence? No. A convergent series may not satisfy (2). For example take the sequence $a_n = \frac{1}{n}$, with limit $L = 0$ and $\varepsilon = 1$. Also every divergent sequence may not satisfy (3). For example consider the sequence $a_n = (-1)^n$, $n \in \mathbb{N}$.

Problem 5. Determine which of the following sequences converge and which diverge. In the case of convergence, determine the limit.

(1) $\left\{\dfrac{2 - 3n^2}{n^2 + 2n + 1}\right\}_{n\in\mathbb{N}}$

Proof. Multiply both numerator and denominator by $\frac{1}{n^2}$ and obtain $\left\{\dfrac{\frac{2}{n^2} - 3}{1 + \frac{2}{n} + \frac{1}{n^2}}\right\}_{n\in\mathbb{N}}$.
Now use limit laws to show that the limit is -3. □

(2) $\left\{\dfrac{3n^2 - 2}{3n + 1}\right\}_{n\in\mathbb{N}}$

Proof. Observe that, for any $n \in \mathbb{P}$, since $3n + 1 \neq 0$,

$$\frac{3n^2 - 2}{3n + 1} = \frac{n(3n + 1) - n - 2}{3n + 1} = n - \frac{n + 2}{3n + 1} \geq n - 1,$$

where in the last step, we use that $\dfrac{n + 2}{3n + 1} < 1$ for all $n \geq 1$.
By the Archimedean property of $\mathbb{R}$/unboundedness of $\mathbb{N}$/divergence of $\{n\}$: given any $M \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > M$. Thus,

$$\frac{3n^2 - 2}{3n + 1} > M.$$

Since $M \in \mathbb{R}$ was arbitrary, the given sequence diverges to ∞. □

(3) $\left\{n-\sqrt{1+n^2}\right\}_{n\in\mathbb{N}}$

Proof. Multiply and divide by $n+\sqrt{1+n^2}$ and then use Archimedean Property to show that the limit is 0. □

(4) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n\in\mathbb{N}}$

Proof. Take $\epsilon=\frac{1}{4}$ and consider a_{2n} and a_{2n+1}. Then proceed using triangular inequality as in the proof of when we showed that $a_n=(-1)^n$, $n\in\mathbb{N}$ is divergent □